# Columns


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COLUMNS

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# COLUMNS

## A TREATISE ON THE STRENGTH AND DESIGN OF COMPRESSION MEMBERS

BY

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# PREFACE

THIS work is approximately one-third of a Treatise on the same subject which the author submitted in 1916 to the University of London as a thesis for the degree of D.Sc. (Engineering). The Thesis, on which the author spent nine years (1906–1915), consisted of three parts: I. Historical; II. Analytical; III. Synthetical. Owing to the conditions now prevailing, it has been found impossible to get the complete work published, and in the present volume the historical portion, which consisted of a short summary of each important memoir, including all the published experimental work, has been replaced by a Bibliography. This Bibliography has been brought up to date, and includes, it is believed, all the more important original work on the subject. Articles of secondary interest and repetitions of work previously published have not been included.

Parts II and III have been reproduced practically as they stood in the original, except that notes have been added, where necessary, to bring the work up to date. In Part II the author has endeavoured to give a perfectly general analysis, leading to the consideration of such particular cases as were suggested by his reading. Some of these are well known, others are new. In particular, he has considered the commonest of all cases in practice, the imperfectly direction-fixed column. The analysis for flat-ended, and especially that for lattice-braced columns will, he hopes, prove of value. Unneces sary mathematical refinement has been avoided, and simple approximations sought for practical use. This work has brought to light a number of interesting new points.

His reasons for the terminology, symbols, and definitions employed are fully set out in the Preface to the Thesis, and are not reproduced here.

In Part III an attempt has been made to collate what has been done on the subject. In this portion of the work the author has endeavoured to sum up in a readable form the teachings of both theory and experiment. References have been inserted freely in the form of Author's name and year of publication. The complete reference can thus be at once obtained from the Bibliography. This part terminates with a practical application and examples.

E. H. SALMON.

LONDON, 1920.

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# DEFINITIONS AND SYMBOLS

## DEFINITIONS

IN A COLUMN:

The *centre line*, or *central axis*, is the longitudinal geometrical central axis of the column. It is the line passing through the centre of area of each right cross section.

The *line of resistance* is the longitudinal axis along which acts the resultant of all the internal forces normal to any right cross section. It is the line passing through the centre of resistance of each right cross section.

The *centre of resistance* is the point at which acts the resultant of all the internal forces normal to any right cross section.

The *principal transverse axes* of any right cross section are the principal geometrical axes passing through the centre of area.

The *principal axes of elasticity* of any right cross section are the principal transverse axes passing through the centre of resistance, about which the moment of stiffness is a maximum and a minimum. They are the principal geometrical axes of a thin lamina of the same shape as the cross section, but of which the thickness is proportional to the modulus of elasticity at every point.

The *neutral surface* is the surface, usually imaginary, on which no stress of any kind exists.

The *neutral line* is the intersection of this surface with the plane of flexure (uniplanar bending).

The *neutral axis* is the transverse axis on which no stress of any kind exists. It is the intersection of the neutral surface with any right transverse plane section.

The *load line*, or line of action of the loads, is the line along which the longitudinal loads on the column are directed.

The *moment of stiffness* is the moment of inertia of the lamina described above about one of the principal axes of elasticity. It is a measure of the bending moment necessary to produce a given change of curvature (see Part II, equation 9).

END CONDITIONS:

*Position-fixed ends*. The ends are held fixed in their original position, but are free to turn in direction.

*Direction-fixed ends*. The ends are held fixed in their original direction, that is to say the inclination of the tangent to the ends of the line of resistance remains constant.

*Position- and direction-fixed ends*. The ends are held fixed both in their original position and direction.

*Free ends*. If the ends be not fixed, but are unrestrained in position or direction, as the case may be, they are denoted as "free."

*Round ends*. The ends of the column are hemispherical in shape, the hemisphere resting on a plane flat surface. (Fig. 44.)

*Flat or square ends*. The ends of the column are made flat and perpendicular to the central axis. (Fig. 44.)

*Hinged ends*. The ends of the column are furnished with pins or balls working in sockets.

*Pointed ends.* The ends of the column are furnished with conical points. (Fig. 45.)

*Flanged ends.* The ends of the column are furnished with flanges or discs made flat and perpendicular to the central axis. (Fig. 44.)

It is to be understood that in all references to practical and experimental work such terms as "straight," "flat," "perpendicular," "concentrically loaded" imply only such a measure of perfection as is possible and usual in practice.

In *concentrically loaded specimens* the load is applied at the centre of area of the end cross sections.

In *eccentrically loaded specimens* the load is applied at a distance from the centre of area of the end cross sections.

## SYMBOLS

A . . = the origin.
= a constant.

B . . = the point of application of the load.
= the breadth or maximum transverse dimension of the cross section.
= a constant.

C . . = a constant.
$C_0$ . . = the centre of area of the cross section.
$CC_0C$ . = the principal transverse axis of the cross section perpendicular to the plane of bending (uniplanar bending).

D . . = the width or minimum transverse dimension of the cross section.
$D_0$ . . = the centre of resistance.
$DD_0D$ . = the principal axis of elasticity passing through the centre of resistance and perpendicular to the plane of bending (uniplanar bending).

E . . = the modulus of elasticity (Young's modulus).
$E_a$ . . = the average value of the modulus of elasticity.
$E_1$ . . = the value of E on the convex side of the column.
$E_2$ . . = the value of E on the concave side of the column.

F . . = a force other than the longitudinal load.
*In lattice-braced columns:*
$F_1$ . = the force in the convex flange.
$F_2$ . = the force in the concave flange.
$F_c$ . = the maximum force in an elementary flange column.

G . . = the modulus of elasticity in shear.

$H_1HH_0H_2$ = a cross section of the column.
$H_1H_2$ . = points on the contour of the cross section.

I . . = the moment of inertia of the cross section about a principal axis.
$I_1$ . . = the moment of inertia of the area $a_1$ about its own centre of area.
$I_2$ . . = the moment of inertia of the area $a_2$ about its own centre of area.

$J_1JJ_0J_2$ . = a cross section of the column.

K . . = a constant.
$K_0$ . . = any point on the line of resistance.
$K_1KK_0K_2$ = a cross section of the column.

L . . = the length of the column.
$L_1$ . . = the original length of the column.

$M$ . . = a bending moment.
$M_a$ and $M_b$ = the direction-fixing moments at the ends of the column.
$M_1$ . . = an accidental bending moment.

N . . = the normal component of the forces on a cross section.

O . . = the centre of curvature, usually of the line of resistance.
$O_1$ . . = the original centre of curvature of the line of resistance.

P . . = Euler's crippling load for a position-fixed column.
$P_2$ . . = Euler's crippling load for a position- and direction-fixed column.
$P_3$ . . = Euler's crippling load for a column with both ends fixed in position and one end fixed in direction.

Q . . = the transverse component of the forces on a cross section.
= a shearing force.

R . . = the ultimate resistance or strength of a column.
$R_0$ . . = the ultimate resistance of a very short column.

S . . = the moment of stiffness of a column.

T . . = the ratio of stress to strain $\frac{df}{ds}$ after the elastic limit has been passed.
$T_1$ . . = the value of T on the convex side.
$T_2$ . . = the value of T on the concave side.

U . . = the ends of the line of resistance.
$UU_0U$ . = the line of resistance.
U . . = the work done in deforming a column.

V . . = the ends of the central axis.
$VV_0V$ . = the central axis.

W . . = the longitudinal load.
W′ . . = the transverse load on a laterally loaded column.

X . . = current co-ordinate.
= the ratio $\frac{L}{\kappa}$
$X_p$ . . = the validity limit of Euler's formula.

Y . . = the total deflection of the column measured from the load line (Fig. 37).
$Y_0$ . . = the maximum value of Y.

Z . . = the modulus of resistance of the cross section.

$a$ . . = area ; the total area of the column.
$a_0$ . . = the area necessary in a short column to sustain the load.
$a_1$ . . = the area of the convex side or flange of the column.
$a_2$ . . = the area of the concave side or flange of the column.

$b$ . . = the width of the cross section.

$c$ . . = a constant or coefficient.
$c_1$ and $c_2$ . = the constants in empirical formulæ.
= the ratio of the contraction of length to the load in lattice-braced columns.
$c_3$ . . = a constant depending on end conditions.

$d$ . . = symbol of differentiation.

$e$ . . = the ratio $\frac{E_1 - E_2}{E_a}$

$f$ . . = stress.
$f_a$ . . = the direct compressive stress $= \frac{dW}{da}$
$f_b$ . . = the stress due to bending.
$f_c$ . . = the maximum compressive stress $= f_a + f_b$.
$f_e$ . . = the stress at the elastic limit.
$f_p$ . . = Euler's crippling load per unit of area $= \frac{P}{a}$.
$f_r$ . . = the ultimate load per unit of area $= \frac{R}{a}$.
$f_s$ . . = the shear stress.
$f_t$ . . = the maximum tensile stress $= f_a - f_b$.
$f_w$ . . = the load per unit of area at which wrinkling occurs.
$f_y$ . . = the stress at the yield point.
$f_C$ . . = the ultimate compressive stress of the material.
$f_T$ . . = the ultimate tensile stress of the material.
$f_1$ . . = the stress where $v = v_1$ and $u = u_1$.
$f_2$ . . = the stress where $v = v_2$ and $u = u_2$.

$g$ . . $= \frac{a}{\kappa^2}$.

$h$ . . = the distance between the centres of area of the flanges in a lattice-braced column.

$j$ . . = the panel length in a lattice-braced column (Fig. 35).

$k$ . . = a constant or coefficient.
= a coefficient expressing the increase in the inclination of the line of resistance at the ends of the column due to the load.

$l$ . . = a length of arc of the line of resistance.

$m$ . . = a constant, coefficient, or index.
= Poisson's ratio.

$n$ . . = a constant, coefficient, or index.

$p$ . . = pitch ; pitch of rivets.

$q$ . . = a coefficient $= \frac{\lambda}{L}$ in originally straight columns. $q$L is the "free length" of the column.

$r$ . . = a constant in Euler's formula depending on the end conditions.
= radius.

$s$ . . = strain.
$s_a$ . . = direct strain $= E \cdot \frac{dW}{da}$.
$s_b$ . . = the strain due to bending.
$s_e$ . . = the strain at the elastic limit.

$t$ . . = thickness.

$u$ . . = the distance of a tube of fibres from the principal axis of elasticity.
$u_1$ . . = the distance of the extreme fibres on the convex side from the principal axis of elasticity.
$u_2$ . . = the distance of the extreme fibres on the concave side from the principal axis of elasticity.
$u_n$ . . = the distance of the neutral axis from the principal axis of elasticity.

$v$ . . = the distance of a tube of fibres from the principal transverse axis.
$v_1$ . . = the distance of the extreme fibres on the convex side from the principal transverse axis.
$v_2$ . . = the distance of the extreme fibres on the concave side from the principal transverse axis.
$v_n$ . . = the distance of the neutral axis from the principal transverse axis.
$v_d$ . . = the distance between the central axis and the line of resistance.
$\bar{v}_1$ . . = the distance of the centre of area $a_1$ from the principal transverse axis.
$\bar{v}_2$ . . = the distance of the centre of area $a_2$ from the principal transverse axis.

$w$ . . = the lateral load per unit run.

$x$ . . = current abscissa.

$y$ . . = the current co-ordinate measured from the load line.
= the deflection of the column measured from the load line (uniplanar bending).
$y_1$ . . = the initial deflection of the column measured from the load line (uniplanar bending).
$y_0$ . . = the maximum value of $y$.
$y$ . . = the component of the total deflection Y parallel to the axis of $y$.

$z$ . . = the component of the total deflection Y parallel to the axis of $z$.
$z_0$ . . = the maximum value of $z$.

$a$ . . $= \sqrt{\frac{W}{S(1-s_a)}}$ which reduces, under ideal conditions, to $\sqrt{\frac{W}{EI}}$. See also Chapter II, equation (301).

$\beta$ . . $= \frac{\epsilon v_2}{\kappa^2}$.

$\delta$ . . = finite difference.
$\delta L$ . . = the total contraction of length in a column.
$\delta' L$ . . = that portion of $\delta L$ due to direct compression.
$\delta'' L$ . . = that portion of $\delta L$ due to bending.
$\epsilon$ . . = an eccentricity or initial deflection.
$\epsilon_1$ . . = the initial deflection.
$\epsilon_2$ . . = the eccentricity of loading.

$\zeta$ . . $= \frac{a}{Q^2}\int f_s^2.da$. A coefficient depending on the distribution of shear stress over the cross section.

$\eta$ . . = the factor of safety.

$\theta$ . . = the angle between a tangent to the line of resistance and the axis of $x$, $\tan\theta = \frac{dy}{dx}$.

$\theta_a$ . . = the value of $\theta$ at the extremities of the column.

$\kappa$ . . = the radius of gyration.

$\lambda$ . . = the semi wave length of the deflection curve.

$\mu$ . . = Tetmajer's empirical coefficient.

$\xi$ . . = a reduction coefficient.

$\pi$ . . = the ratio of circumference to diameter in a circle.

$\rho$ . . = the radius of curvature of the line of resistance.
$\rho_1$ . . = the original radius of curvature of the line of resistance.

$\sigma$ . . = slope $= \frac{dy}{dx}$.

$\phi$ . . = an angle, particularly in non-uniplanar bending.

$\omega$ . . = the radius of the core.
$\omega_1$ . . = the radius of the core for the convex side of the column.
$\omega_2$ . . = the radius of the core for the concave side of the column.

$\Delta$ . . = the deflection of the centre of the column measured from its original position.
$= y_0 - \epsilon_1 - \epsilon_2$ (uniplanar bending).

$\Omega$ . . = the unsupported width of plate (Fig. 52).

The above symbols are of general applicability. In certain cases, however, different writers have attached slightly different meanings to the symbols. Such variations are always noted in the context and usually denoted by the addition of a suffix or dash.

In the case of *non-uniplanar bending* the suffix $y$ indicates that the symbol has reference to bending in the plane $xy$. The suffix $z$ or the addition of a single dash (e.g. $v_2'$) indicates that the symbol has reference to bending in the plane $xz$.

In the case of *lattice-braced columns* the addition of two dashes or three dashes indicates that the symbol has reference to secondary or tertiary flexure respectively.

# COLUMNS

## PART I

## *BIBLIOGRAPHY*

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# PART II

## *ANALYTICAL*

### CHAPTER I

### *GENERAL FORMULÆ FOR SOLID COLUMNS. UNIPLANAR BENDING*

IN establishing these formulæ it will be assumed:

1. That the limit of elasticity is not exceeded, and that the stress is proportional to the strain.

2. That the bending is uniplanar, that is to say that it takes place in one plane only.

3. That the curvature is always small, and that the ordinary theory of bending may be applied.

4. That the modulus of elasticity is different at different points in the column.

5. That the column acts as a whole, and that no "secondary flexure" or local deformation of the separate parts of the member takes place.

6. That the weight of the column itself may be neglected. This only becomes of importance in practical columns when it forms a lateral load on the member (see chapter ii).

*The General Case.*—Let UU, Fig. 1, be a column acted on by longitudinal forces W and external bending moments $M_a$, which cause it to shorten and bend.

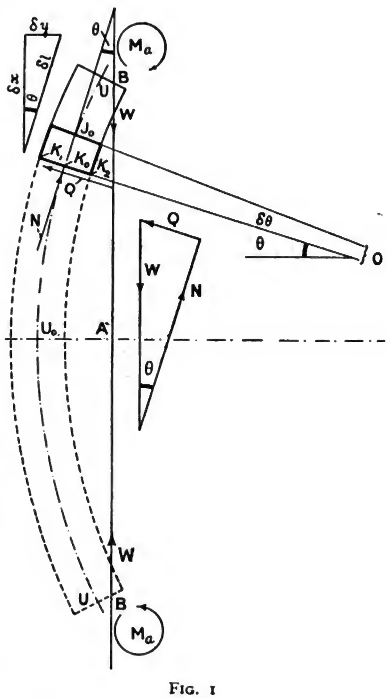


FIG. 1

2

Let $K_0U$ be a portion of the column cut off by the normal section $K_1K_0K_2$, and O be the centre of curvature of the line of resistance of the column at the point $K_0$.

Now, whatever be the internal stresses set up at the section $K_1K_0K_2$, they will reduce to a normal component N, a tangential component Q, and a moment, the bending moment M on the cross section.

But both the moments and the forces acting on the portion $K_0U$ must be in equilibrium. That is to say, the moment M must be equal to the sum of the moment due to the force W and the external moment $M_a$, and further, the three forces W, N, and Q must be proportional to the sides of the triangle of forces shown in the figure.

Let the direction of W be the axis of $x$, and let the axis of $y$ be perpendicular to it. Assume that bending takes place solely in the plane $xy$. Let $\delta l$ be a small element of the length of the column in its strained condition. Then the triangles W, N, Q and $\delta l$, $\delta x$, $\delta y$ will be similar, and therefore

$$Q = W \cdot \frac{dy}{dl} = W \sin \theta$$

$$N = W \cdot \frac{dx}{dl} = W \cos \theta$$

where $\theta$ is the angle which the line $K_1K_0K_2$ makes with the axis of $y$.

From tables of experimental results it will be seen that, for practical columns, the value of $y_0$, the maximum deflection, even at the maximum load, is exceedingly small; and therefore the value of $\theta$ will be very small indeed, and $\cos \theta$ will be practically equal to unity. No sensible error will therefore be introduced, and much needless complication will be saved, if N be assumed equal to W. For similar reasons the effect of the shearing force Q may be neglected in all solid columns, a conclusion reached by every investigator who has included the effect of Q in his analysis. Built-up members require special consideration.

*The Stresses on a Cross Section.*—Fig. 2 is an enlarged view of the elementary length $J_0K_0$. Suppose that $H_1H_2K_2K_1$ was its original shape before distortion, $J_1J_2K_2K_1$ being its shape when under strain. Let $J_0K_0$ be a layer of fibres unaltered in length by the *bending moment*, that is to say, in the layer $J_0K_0$ no stress due to the bending moment and therefore pure compression only exists. Assume that normal sections which were plane before distortion remain plane and normal afterwards. Then O, the intersection of the two normal boundary planes $J_1J_0J_2$ and $K_1K_0K_2$ of the element $J_0K_0$, will be the centre of curvature of the distorted element, or, strictly speaking, of the layer of fibres $J_0K_0$. Let the radius of curvature $OK_0 = \rho$.

Now the original shape of the line $J_0K_0$ was $H_0K_0$, and the plane $J_1J_0J_2$ was originally the plane $H_1H_0H_2$. Hence $O_1$ was the original centre of curvature of the element, and $O_1K_0$ the original radius of curvature. Let $O_1K_0 = \rho_1$. It will be presumed that the initial curvature is small, that is to say, that $\rho_1$ is large. Let the angle $K_0OJ_0$ be $\delta\theta$ and the angle $K_0O_1H_0$ be $\delta\theta_1$.

Suppose $DD_0D$ to be the intersection of the layer $J_0K_0$ with the plane $K_1K_0K_2$. Then $DD_0D$ will be perpendicular to $OD_0$, the trace of the plane of flexure on the plane $K_1K_0K_2$. Consider a small tube of fibres JK distant $u$ from the line $DD_0D$, and suppose that HK was the length and shape of the tube previous to distortion.

Then the positive alteration of length (elongation) of this tube of fibres due to the load and the bending moment is

$$(\rho + u)\delta\theta - (\rho_1 + u)\delta\theta_1$$

and the positive strain is

$$\frac{(\rho + u)\delta\theta - (\rho_1 + u)\delta\theta_1}{(\rho_1 + u)\delta\theta_1}$$

$$= \frac{(\rho + u)\dfrac{\delta\theta}{\delta\theta_1} - (\rho_1 + u)}{(\rho_1 + u)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Now } \frac{\delta\theta}{\delta\theta_1} = \frac{J_0K_0}{H_0K_0} \cdot \frac{\rho_1}{\rho}$$

$$= \left(1 - \frac{H_0K_0 - J_0K_0}{H_0K_0}\right)\frac{\rho_1}{\rho}.$$

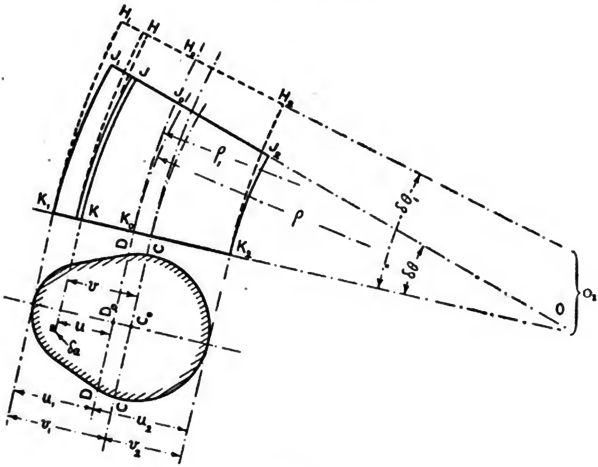


FIG. 2

But $J_0K_0$ is the strained length of the layer of fibres originally $H_0K_0$ in length, and this layer is unaltered in length by the bending moment. Therefore the strain in it $\dfrac{H_0K_0 - J_0K_0}{H_0K_0}$ is solely due to the direct compressive stress produced by W, and is independent of the value of M, the bending moment. It is, therefore, the strain which would exist if the bending moment were zero, and W produced pure compression only. Since plane sections remain plane sections after distortion of the element, it is evident that, if $M = O$, this strain would be uniform all over the cross section, and therefore the strain in the tube of fibres JK would be equal to the strain in the layer $J_0K_0$. Call this strain $s_a$. Let $\delta W$ be the proportion of W carried by the tube of fibres JK under these conditions ($M = O$), and let $\delta a$ be the area of cross

section of that tube. Then if E be the modulus of elasticity of the fibres composing the tube, $\frac{\delta W}{\delta a}$ is the stress in the tube, and the strain

$$s_a = \frac{1}{E} \cdot \frac{\delta W}{\delta a} \quad \ldots \quad (2)$$

Hence the strain in the layer $J_0K_0$, and therefore the fraction

$$\frac{H_0K_0 - J_0K_0}{H_0K_0} = \frac{1}{E} \cdot \frac{\delta W}{\delta a}$$

from which

$$\frac{\delta\theta}{\delta\theta_1} = \left(1 - \frac{1}{E} \cdot \frac{\delta W}{\delta a}\right) \frac{\rho_1}{\rho} \quad \ldots \quad (3)$$

The positive strain $s$ in the tube of fibres JK when both W and the bending moment M act together is, therefore, from equations (1) and (3)

$$s = \frac{(\rho + u)\left(1 - \frac{1}{E} \cdot \frac{\delta W}{\delta a}\right)\frac{\rho_1}{\rho} - (\rho_1 + u)}{(\rho_1 + u)}$$

$$= \frac{u(\rho_1 - \rho) - \frac{1}{E} \cdot \frac{\delta W}{\delta a}(\rho + u)\rho_1}{\rho(\rho_1 + u)}$$

$$= u\frac{\rho_1 - \rho}{\rho(\rho_1 + u)}\left(1 - \frac{1}{E} \cdot \frac{\delta W}{\delta a}\right) - \frac{1}{E} \cdot \frac{\delta W}{\delta a}.$$

But since the initial curvature is assumed very small, $\rho_1$ will be very large, and $u$ in the denominator may be neglected in comparison with it. Hence

$$s = u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{1 - \frac{1}{E} \cdot \frac{\delta W}{\delta a}\right\} - \frac{1}{E} \cdot \frac{\delta W}{\delta a} \quad \ldots \quad (4)$$

If $f$ be the intensity of the tensile stress in the tube of fibres,

$$f = u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{E - \frac{\delta W}{\delta a}\right\} - \frac{\delta W}{\delta a} \quad \ldots \quad (5)$$

and the tensile load on the tube is

$$f \cdot \delta a = u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{E - \frac{\delta W}{\delta a}\right\}\delta a - \frac{\delta W}{\delta a} \cdot \delta a.$$

The total tensile force on the cross section is, therefore,

$$\int_{u=u_2}^{u=u_1} f \cdot da = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right)u \cdot da - \int_{u_2}^{u_1}\frac{dW}{da} \cdot da.$$

But the total tensile force on the cross section is $-$ W. Hence

$$-W = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right)u \cdot da - \int_{u_2}^{u_1}\frac{dW}{da} \cdot da.$$

Now $\int_{u_2}^{u_1}\frac{dW}{da} \cdot da = W$. Therefore $\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right)u \cdot da = 0$;

and hence
$$\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right) u \cdot da = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But from equation (2) $\frac{dW}{da} = Es_a$. Therefore $(1 - s_a)\int_{u_2}^{u_1} E \cdot u \cdot da = 0$.

Now $(1 - s_a)$ cannot be zero unless $s_a = 1$, which is impossible. Therefore
$$\int_{u_2}^{u_1} E \cdot u \cdot da = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This relation determines the position of the line $DD_0D$, and thus the layer $J_0K_0$.

From equation (6) $\int_{u_2}^{u_1} E \cdot u \cdot da = \int_{u_2}^{u_1} \frac{dW}{da} \cdot u \cdot da$. Hence, from equation (7)
$$\int_{u_2}^{u_1} \frac{dW}{da} \cdot u \cdot da = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $u_3$ be the distance of the line of action of the resultant of the normal forces on the cross section from the layer of fibres $J_0K_0$, then
$$\int_{u_2}^{u_1} \frac{dW}{da} \cdot u \cdot da = Wu_3 = 0.$$

Hence $u_3 = 0$. That is to say the resultant W of the normal forces on the cross section acts along the line $J_0K_0$. The layer of fibres $J_0K_0$ would be the neutral surface of the member supposing it to be subjected to a bending moment only, for equation (7) is the relation determining the neutral axis in a beam.

Since the bending is *ex hypothesi* uniplanar, it is evident that the lines $OD_0$ and $DD_0D$ are the principal *axes of elasticity* of the cross section, and $D_0$ is the *centre of resistance* or the point at which the resultant W acts. These principal axes of elasticity are analogous to the principal geometric axes, but not, in general, coincident with them. The point $D_0$ is evidently the centre of gravity of a lamina of the shape of the cross section, but of which the thickness is proportional to the modulus of elasticity at every point. If the value of E were known everywhere, the co-ordinates of $D_0$ and the direction of the principal axes of elasticity might be found graphically for any given shape of cross section.

The line joining all such points as $D_0$ may be called the *line of resistance* of the column, for the direction of the resultant W (or, strictly speaking, N) is always tangent to it. The line $J_0K_0$ is the elevation of the line of resistance.

*The Moment of Resistance.*—Take moments about the line $DD_0D$. About this line the moment of the resultant W will be zero, and the moment of the stresses will form a couple.

The moment of the force on an elementary area $\delta a$
$$= f \, . \, u \, . \, \delta a$$
or, from equation (5),
$$= u^2\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{E - \frac{\delta W}{\delta a}\right\}\delta a - u\frac{\delta W}{\delta a} \cdot \delta a.$$

Hence the moment of resistance

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right)u^2 \cdot da - \int_{u_2}^{u_1} u \cdot \frac{dW}{da} \cdot da,$$

which reduces to [see equations (8) and (2)]

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\int_{u_2}^{u_1}\left(E - \frac{dW}{da}\right)u^2 \cdot da$$

$$= \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a)\int_{u_2}^{u_1} Eu^2 \cdot da.$$

Now the integral $\int_{u_2}^{u_1} E\, u^2 \cdot da$ is the moment of inertia of the lamina described above, about the principal axis $DD_0D$ ($u = O$). It might be termed the *moment of elasticity* or, to revive Euler's expression, the *moment of stiffness* of the column, since it is a measure of the bending moment necessary to produce a given change of curvature. Call, therefore,

$$\int_{u_2}^{u_1} Eu^2 \cdot da = S.$$

Then
$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a)S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

*The Neutral Surface.*—The neutral surface may be obtained by putting $f = O$. From equations (5) and (2)

$$f = E\left\{u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a) - s_a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When $f = O$, let $u = u_n$, then

$$u_n = \frac{\rho\rho_1}{\rho_1 - \rho} \cdot \frac{s_a}{1 - s_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which determines the position of the neutral surface.

For completeness it may be well to show that the exact equations obtained with the Bernoulli-Euler assumptions follow from the above equations. If the column be originally straight and homogeneous (E constant), $s_a = \frac{W}{Ea}$, or more strictly

$$s_a = \frac{N}{Ea} = \frac{W}{Ea} \cdot \frac{dx}{dl}.$$

The principal axes of elasticity will coincide with the principal geometric axes, and $u_n = v_n$. The original curvature $\frac{1}{\rho_1} = O$, and hence from equation (11)

$$\frac{\rho}{v_n} = \frac{1 - s_a}{s_a}, \quad \frac{\rho + v_n}{v_n} = \frac{1}{s_a},$$

and
$$\frac{v_n Ea}{\rho + v_n} = W\frac{dx}{dl} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From equation (9)

$$M = \frac{1}{\rho}(1 - s_a) S$$

$$M = \frac{S}{\rho + v_n} = \frac{EI}{\rho + v_n} \quad \dots\dots\dots \quad (13)$$

Eliminating $v_n$ between equations (12) and (13)

$$\frac{S}{\rho}\left[1 - \frac{W}{Ea} \cdot \frac{dx}{dl}\right] = M \quad \dots\dots\dots \quad (14)$$

where $S = EI$. Compare Lamarle (1846) and others.

*The Shape of the Bent Column.*—Let $UU_0U$ (Fig. 1) be the line of resistance of the bent column, $K_0$ any point thereon. Let A in the line of action of the load be taken as origin, AB be the axis of $x$ and $AU_0$, at right angles thereto, the axis of $y$. Let $x$ and $y$ be the co-ordinates of $K_0$, where $y$ is the deflection of the line of resistance from the load line. Then if M be the bending moment at $K_0$, from equation (9)

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a)S.$$

Suppose the co-ordinates of $K_0$ in its unstrained position to have been $x$ and $y_1$. Then since the curvature both before and after bending is very small

$$\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = -\frac{d^2}{dx^2}(y - y_1).$$

(The curve is concave towards the axis of $x$.) Hence

$$\frac{d^2}{dx^2}(y - y_1) + \frac{M}{S(1 - s_a)} = 0 \quad \dots\dots\dots \quad (15)$$

a differential equation the solution to which gives the shape of the bent line of resistance.

*The Stresses in the Extreme Fibres.*—From equations (9) and (10)

$$f = E\left\{u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a) - s_a\right\}$$

and

$$\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a) = \frac{M}{S}. \quad \text{Hence}$$

$$f = E\left\{\frac{Mu}{S} - s_a\right\} \quad \dots\dots\dots \quad (16)$$

The stress in the extreme fibres on the tension side of any cross section where $u = u_1$ and $E = E_1$ is

$$f_1 = E_1\left\{\frac{Mu_1}{S} - s_a\right\} \quad \dots\dots\dots \quad (17)$$

and in the extreme fibres on the compression side where $u = -u_2$ and $E = E_2$ is

$$f_2 = -E_2\left\{\frac{Mu_2}{S} + s_a\right\} \quad \dots\dots\dots \quad (18)$$

the negative sign denoting compression. These stresses are not necessarily the maximum stresses on the cross section.

*Recapitulation.*—The strain due to the direct compressive action of the load

$$s_a = \frac{1}{E} \cdot \frac{dW}{da} \quad \dots\dots\dots \quad (2)$$

The stress anywhere

$$f = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{E - \frac{dW}{da}\right\}u - \frac{dW}{da} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

$$= E\left\{u\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a) - s_a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

$$= E\left\{\frac{Mu}{S} - s_a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

The stresses in the extreme fibres

$$f_1 = E_1\left\{\frac{Mu_1}{S} - s_a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

$$f_2 = -E_2\left\{\frac{Mu_2}{S} + s_a\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

To determine the position of the line of resistance

$$\int_{u_2}^{u_1} E \cdot u \cdot da = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The moment of resistance

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)(1 - s_a)S \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

To determine the neutral surface

$$u_n = \frac{\rho\rho_1}{\rho_1 - \rho} \cdot \frac{s_a}{1 - s_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

To determine the shape of the line of resistance

$$\frac{d^2}{dx^2}(y - y_1) + \frac{M}{S(1 - s_a)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

## CASE I. Position-fixed Columns—Uniplanar Bending

*Both ends fixed in position but free in direction*

### VARIATION I. IDEAL CONDITIONS

The column is of uniform cross section and originally perfectly straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and in the direction of the unstrained central axis. Suppose the column to bend.

Since E is constant, it follows from equation (2) that

$$\frac{dW}{da} = Es_a = \frac{W}{a}, \text{ and } s_a = \frac{W}{Ea}.$$

Further, the centre of resistance will coincide with the centre of area of the cross section, and the central axis will be the line of resistance. The moment of stiffness S will be constant and equal to EI, where I is the least moment of inertia of the cross section. Since the column was originally straight, the initial curvature $\frac{1}{\rho_1}$ will be zero.

Let $UU_0U$, Fig. 3, be the shape of the bent line of resistance (the bent

central axis of the column). Take origin at A in the line of action of W, $AU = \frac{L}{2}$. If the co-ordinates of any point $K_0$ on the line of resistance be $x$ and $y$, the bending moment there is $M = Wy$.

From equation (15), therefore,

$$\frac{d^2y}{dx^2} + \frac{Wy}{I\left(E - \frac{W}{a}\right)} = 0.$$

Let $\frac{W}{I\left(E - \frac{W}{a}\right)} = \alpha^2$, then

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

to which the solution is

$$y = m \sin \alpha x + n \cos \alpha x.$$

Now, from the symmetry of the figure, when $x = 0$, $\frac{dy}{dx} = 0$, and therefore $m = 0$. Hence, $y = n \cos \alpha x$.

When $x = 0$, $y = n$. Hence $n$ is the deflection at the origin, that is to say, the maximum deflection of the column. Let $n = y_0$, the equation to the bent line of resistance is then

$$y = y_0 \cos \alpha x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

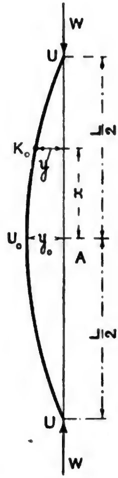

FIG. 3.

Now, when $x = \frac{L}{2}$, $y = 0$. Therefore if $y_0$ have a real value,

$$\cos \frac{\alpha L}{2} = 0.$$

or $\frac{\alpha L}{2} = \frac{r\pi}{2}$. For the practical case, when the column has no points of inflexion, $r = 1$, and

$$\alpha^2 = \frac{\pi^2}{L^2} = \frac{W}{I\left(E - \frac{W}{a}\right)},$$

or

$$W = \frac{\frac{\pi^2 EI}{L^2}}{\left(1 + \frac{\pi^2 I}{a L^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the formula reached by Lamarle (1846).

The value of $\frac{W}{a}$ will be very small compared with E, and may be neglected, in which case

$$W = \frac{\pi^2 EI}{L^2} = P \quad \text{Euler's value.}$$

Under the conditions assumed there is, of course, no reason why the column

should bend at all, unless disturbed by some extraneous influence; and the value of W given by the formula is the least value of the load under which the column will remain bent if so disturbed.

The value of L in the formula is really the length of the chord UU. For all practical purposes it can be put equal to the original length $L_1$ of the column.

Since the proof is based on the elastic theory it follows that the formulæ cease to be valid when the elastic limit is exceeded. Hence they must cease to hold when $W > f_e a$.

### VARIATION 2. COLUMN WITH INITIAL CURVATURE.

The column is of uniform cross section, but not originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and along the line joining its points of application. The effect of the load will be to increase the curvature everywhere.

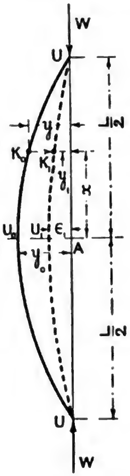


FIG. 4.

Then, as in Variation 1, $s_a = \dfrac{W}{Ea}$, $S = EI = \text{const.}$, the central axis will be the line of resistance, and I is the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending.

Let $UU_1U$, $UU_0U$, Fig. 4, be the initial and final positions of the line of resistance. Take origin at A. $AU = \dfrac{L}{2}$.

Let the co-ordinates of any point $K_0$ in the line of resistance in its final position be $x$ and $y$. Let $K_1$ be the position of the point $K_0$ before the load was applied, and $x$ and $y_1$ its co-ordinates.

Then the bending moment at the point $K_0$ is $M = Wy$, and from equation (15)

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy}{I\left(E - \dfrac{W}{a}\right)} = 0 \quad . \quad . \quad . \quad (22)$$

Three assumptions will be made as to the shape of the curve $UU_1U$.

(*a*) that the curvature is circular
(*b*) ,, ,, ,, ,, parabolic
(*c*) ,, ,, ,, ,, sinusoidal

*Assumption* (*a*).—That the initial curvature is circular. In this case

$$\frac{1}{\rho_1} = \text{constant.}$$

Let $AU_1$, the initial deflection of the line of resistance at the origin, be $\epsilon_1$. Then $UU_1U$ being an arc of a circle,

$$\epsilon_1 (2\rho - \epsilon_1) = \frac{L^2}{4},$$

or
$$\frac{1}{\rho_1} = \frac{8\epsilon_1}{L^2 + 4\epsilon_1^2} = -\frac{d^2 y_1}{dx^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Equation (22) becomes, therefore,

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2 + 4\epsilon_1^2} + a^2y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $a = \dfrac{W}{I\left(E - \dfrac{W}{a}\right)}$ as before.

Let $$a^2Y = a^2y + \frac{8\epsilon_1}{(L^2 + 4\epsilon_1^2)}.$$

Then $$\frac{d^2Y}{dx^2} + a^2Y = 0,$$

to which the solution is

$$Y = m \sin ax + n \cos ax.$$

When $x = 0$, $\dfrac{dY}{dx} = \dfrac{dy}{dx} = 0$, and $m = 0$; hence, $y = n \cos ax - \dfrac{8\epsilon_1}{a^2(L^2 + 4\epsilon_1^2)}$.

When $x = \dfrac{L}{2}$, $y = 0$, and $n = \dfrac{8\epsilon_1}{a^2(L^2 + 4\epsilon_1^2)} \sec \dfrac{aL}{2}$;

hence, $$y = \frac{8\epsilon_1}{a^2(L^2 + 4\epsilon_1^2)}\left\{\sec \frac{aL}{2} \cos ax - 1\right\} \quad . \quad . \quad . \quad . \quad (25)$$

The maximum value $y_0$ of $y$ occurs when $x = 0$.

$$y_0 = \frac{8\epsilon_1}{a^2(L^2 + 4\epsilon_1^2)}\left\{\sec \frac{aL}{2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

*Assumption* (*b*).—That the initial curvature is parabolic. If $AU_1 = \epsilon_1$, then the equation to the line $UU_1U$ is

$$y_1 = \epsilon_1\left\{1 - \frac{4x^2}{L^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and $$\frac{d^2y_1}{dx^2} = -\frac{8\epsilon_1}{L^2}.$$

Equation (22) becomes, therefore,

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + a^2y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

to which the solution is, as in Assumption (*a*),

$$y = \frac{8\epsilon_1}{a^2L^2}\left\{\sec \frac{aL}{2} \cos ax - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and $$y_0 = \frac{8\epsilon_1}{a^2L^2}\left\{\sec \frac{aL}{2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Now $a^2 = \dfrac{W}{I\left(E - \dfrac{W}{a}\right)}$. Neglecting as before $\dfrac{W}{a}$ in comparison with E, $a^2L^2 = \dfrac{WL^2}{EI} = \dfrac{\pi^2W}{P}$. Hence, $\dfrac{aL}{2} = \dfrac{\pi}{2}\sqrt{\dfrac{W}{P}}$. Equation (30) may therefore be written

$$y_0 = \frac{8P}{\pi^2W}\epsilon_1\left\{\sec \frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The maximum compressive stress $f_c$ will occur at the middle of the column

on the concave side. Here $y = y_0$ and $M = Wy_0$. Hence, from equation (18), neglecting the minus sign,

$$f_c = E\left\{\frac{Wy_0u_2}{EI} + \frac{W}{Ea}\right\}.$$

Inserting the value of $y_0$ from equation (30), and putting $u_2 = v_2$,

$$f_c = \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_2(Ea - W)}{WL^2}\left(\sec\frac{\alpha L}{2} - 1\right)\right\} \quad . \quad . \quad . \quad (32)$$

or very nearly $$f_c = \frac{W}{a}\left\{1 + \frac{\epsilon_1 v_2}{\kappa^2}\cdot\frac{8P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\} \quad . \quad . \quad . \quad (33)$$

*Assumption* (*c*).—That the initial curvature is sinusoidal. If $AU_1 = \epsilon_1$, the equation to the line $UU_1U$ is

$$y_1 = \epsilon_1 \cos\frac{\pi}{L}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and $$\frac{d^2y_1}{dx^2} = -\frac{\pi^2}{L^2}\epsilon_1\cos\frac{\pi}{L}x.$$

Equation (22) becomes, therefore,

$$\frac{d^2y}{dx^2} + \frac{\pi^2\epsilon_1}{L^2}\cos\frac{\pi}{L}x + \alpha^2 y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

from which $$y = \frac{\pi^2\epsilon_1}{\pi^2 - \alpha^2L^2}\cos\frac{\pi x}{L} = \frac{\epsilon_1}{1 - \dfrac{WL^2}{\pi^2 I\left(E - \dfrac{W}{a}\right)}}\cos\frac{\pi x}{L} \quad . \quad . \quad . \quad (36)$$

The maximum value of $y$ occurs when $x = 0$, and is

$$y_0 = \frac{\epsilon_1}{1 - \dfrac{WL^2}{\pi^2 I\left(E - \dfrac{W}{a}\right)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

or, neglecting $\dfrac{W}{a}$ in comparison with E,

$$y_0 = \frac{\epsilon_1}{1 - \dfrac{W}{P}} = \frac{P}{P - W}\epsilon_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The maximum compressive stress $f_c$ at the middle of the column on the concave side is, by equation (18),

$$f_c = E\left\{\frac{Wy_0u_2}{EI} + \frac{W}{Ea}\right\} = \frac{W}{a}\left\{1 + \frac{\epsilon_1 v_2}{\kappa^2}\,\frac{1}{1 - \dfrac{WL^2}{\pi^2 I\left(E - \dfrac{W}{a}\right)}}\right\} \quad . \quad (39)$$

or very nearly $$f_c = \frac{W}{a}\left\{1 + \frac{\epsilon_1 v_2}{\kappa^2}\cdot\frac{P}{P - W}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

To show that these three assumptions (*a*) (*b*) and (*c*) give practically identical results, the maximum deflection produced in each case will be compared.

(*a*) *Circular*

$$y_0 = \frac{8\epsilon_1}{\alpha^2(L^2 + 4\epsilon_1^2)}\left\{\sec\frac{\alpha L}{2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

(*b*) *Parabolic* :

$$y_0 = \frac{8\epsilon_1}{a^2L^2}\left\{\sec\frac{aL}{2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$= \frac{8\epsilon_1 P}{\pi^2 W}\left\{\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(*c*) *Sinusoidal* :

$$y_0 = \frac{\epsilon_1}{1 - \frac{a^2L^2}{\pi^2}} = \frac{P}{P - W}\epsilon_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Comparing (*a*) with (*b*), it is evident that the two formulæ give practically the same results, for $(L^2 + 4\epsilon_1^2)$ is very nearly equal to $L^2$, since $\epsilon_1$ will be very small compared with L. The parabolic arc would deflect slightly more than the circular arc. Comparing (*b*) with (*c*), and using the two approximate expressions,

*Parabolic*

$$\frac{y_0}{\epsilon_1} = \frac{8P}{\pi^2 W}\left\{\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

*Sinusoidal* :

$$\frac{y_0}{\epsilon_1} = \frac{1}{1 - \frac{W}{P}} = \frac{P}{P - W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The following table, columns 6 and 7, gives the values of $\frac{y_0}{\epsilon_1}$ for varying values of $\frac{W}{P}$. It will be seen that the figures are almost identical. These values are plotted also in Fig. 5. The number on the curve corresponds to its equation number.

It may be concluded from these figures and curves that at any rate for small initial deflections the initial shape makes little difference to the result.

| $\frac{W}{P}$ | $\sqrt{\frac{W}{P}}$ | $\frac{\pi}{2}\sqrt{\frac{W}{P}}$ | $\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}$ | $\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1$ | $\frac{8}{\pi^2}\cdot\frac{P}{W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)$ | $\frac{P}{P-W}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0° 0′ | 1·0000 | ·0000 | 1·0000 | 1·000 |
| ·1 | ·3162 | 28 27½ | 1·1374 | ·1374 | 1·1137 | 1·111 |
| ·2 | ·4472 | 40 15 | 1·3102 | ·3102 | 1·2572 | 1·250 |
| ·3 | ·5477 | 49 17½ | 1·5333 | ·5333 | 1·4409 | 1·429 |
| ·4 | ·6325 | 56 55½ | 1·8324 | ·8324 | 1·6868 | 1·667 |
| ·5 | ·7071 | 63 38½ | 2·2523 | 1·2523 | 2·0302 | 2·000 |
| ·6 | ·7746 | 69 43 | 2·8846 | 1·8846 | 2·5460 | 2·500 |
| ·7 | ·8367 | 75 18 | 3·9408 | 2·9408 | 3·4053 | 3·333 |
| ·8 | ·8944 | 80 30 | 6·0589 | 5·0589 | 5·1257 | 5·000 |
| ·9 | ·9487 | 85 23 | 12·4241 | 11·4241 | 10·289 | 10·000 |
| 1·0 | 1·0000 | 90 0 | ∝ | ∝ | ∝ | ∝ |

### VARIATION 3. THE ECCENTRICALLY LOADED COLUMN

The column is of uniform cross section, and originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at a distance $\epsilon_2$ from the centre of area of

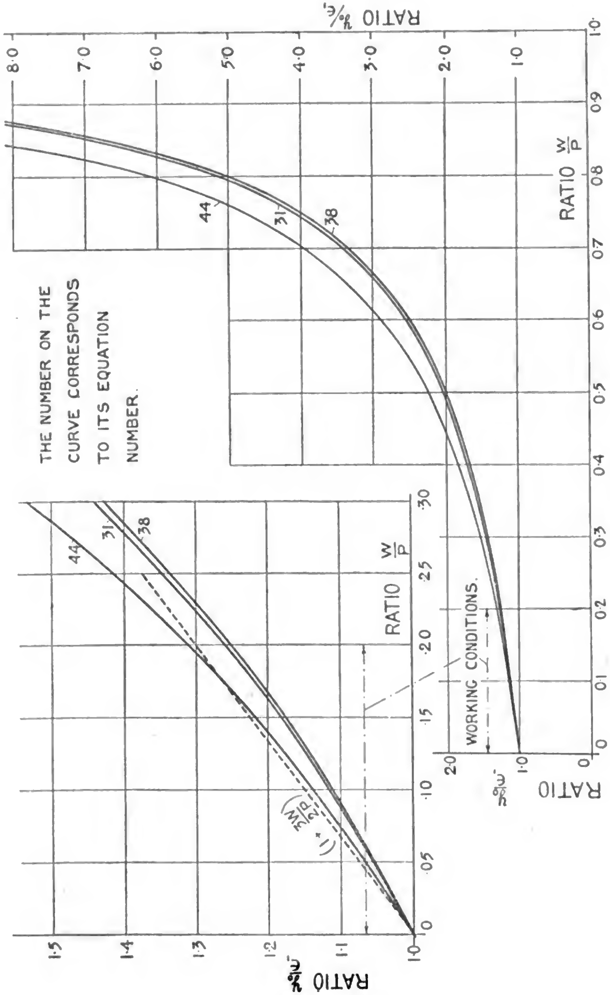
THE NUMBER ON THE CURVE CORRESPONDS TO ITS EQUATION NUMBER.
RATIO $\frac{W}{P}$
RATIO $\frac{W}{P}$
WORKING CONDITIONS.
44
31
38
44
31
38
$\left(1+\frac{3W}{2P}\right)$

FIG. 5.

the cross section at each end, the points of application lying on the same principal axis and on the same side of the central axis, at each end. Owing to this eccentricity the load will cause the column to bend.

Then, as in Variation 1, $s_a = \frac{W}{Ea}$, $S = EI = \text{const.}$, the central axis will be the line of resistance, and I is the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending. The initial curvature $\frac{1}{\rho_1}$ will be zero.

Let $UU_0U$ be the shape of the line of resistance of the bent column, Fig. 6, and BB the points of application of the load. $UB = \epsilon_2$. Take origin at A. $AB = \frac{L}{2}$. Let the co-ordinates of any point $K_0$ in the line of resistance be $x$ and $y$. Then the bending moment at the point $K_0$ is $M = Wy$, and from equation (15)

$$\frac{d^2y}{dx^2} + \frac{Wy}{I\left(E - \frac{W}{a}\right)} = 0 \quad . \quad . \quad . \quad . \quad (41)$$

Calling as before $$a^2 = \frac{W}{I\left(E - \frac{W}{a}\right)}$$

$$\frac{d^2y}{dx^2} + a^2y = 0,$$

to which the solution is $y = m \sin ax + n \cos ax$.

When $x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$. When $x = \frac{L}{2}$, $y = \epsilon_2$, and $n = \epsilon_2 \sec \frac{aL}{2}$. Hence the equation to the line of resistance is

$$y = \epsilon_2 \sec \frac{aL}{2} \cos ax \quad . \quad . \quad . \quad . \quad . \quad (42)$$

When $x = 0$, the maximum value of $y$ is

$$y_0 = \epsilon_2 \sec \frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

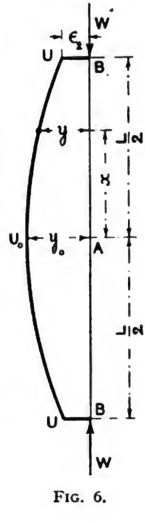


FIG. 6.

or with the same approximations as before,

$$y_0 = \epsilon_2 \sec \frac{\pi}{2}\sqrt{\frac{W}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

The actual deflection produced by the load is

$$\Delta = y_0 - \epsilon_2 = \epsilon_2\left(\sec \frac{aL}{2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The maximum compressive stress $f_c$ occurs at the middle of the column on the concave side. From equation (18)

$$f_c = E\left\{\frac{Wy_0u_2}{EI} + \frac{W}{Ea}\right\} = \frac{W}{a}\left\{1 + \frac{v_2\epsilon_2}{\kappa^2} \sec \frac{aL}{2}\right\} \quad . \quad . \quad . \quad (46)$$

or very nearly $$f_c = \frac{W}{a}\left\{1 + \frac{v_2\epsilon_2}{\kappa^2} \sec \frac{\pi}{2}\sqrt{\frac{W}{P}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

From equation (44) the ratio

$$\frac{y_0}{\epsilon_2} = \sec\frac{\pi}{2}\sqrt{\frac{W}{P}},$$

the values of which are given in the fourth column of the table on p. 29, and are plotted in Fig. 5. It will be evident that for a given value of $\frac{W}{P}$ the ratio $\frac{y_0}{\epsilon_2}$ is larger in an eccentrically loaded column than in one with an initial deflection. An eccentricity of loading, therefore, produces larger deflections and stresses than an equal initial deflection.

## VARIATION 4. THE NON-HOMOGENEOUS COLUMN

The column is of uniform cross section and originally straight. The load is applied at the centres of area of the end cross sections, and in the direction of the unstrained central axis. The modulus of elasticity will be assumed to vary. In practice this variation will not, in general, follow any particular law, but in order to treat the question mathematically, three assumptions will be made.

*Assumption* (*a*).—That the material on the concave side of the central axis has a different modulus of elasticity to the material on the convex side.

*Assumption* (*b*).—That the modulus of elasticity varies uniformly across the column.

*Assumption* (*c*).—That the modulus of elasticity varies both in the direction of the width and the length of the column.

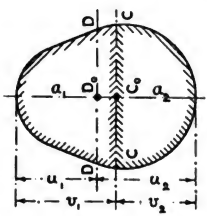

FIG. 7.

*Assumption* (*a*).—That the material on the concave side of the column has a modulus of elasticity $E_2$, and that on the convex side a modulus $E_1$.

Let Fig. 7 represent any cross section of the column. $CC_0C$ is the principal geometric axis perpendicular to the plane of bending. Then it will be assumed that to the right of the line $CC_0C$ the modulus is $E_2$, and to the left its value is $E_1$.

The centre of resistance $D_0$ will no longer coincide with the centre of area, but lie at a distance $\epsilon$ from it. One principal axis of elasticity will coincide with one principal geometric axis, and both will lie in the plane of bending. Hence the curvature will still be uniplanar, and the other two axes, $CC_0C$ and $DD_0D$, will both be perpendicular to the plane of bending.

Let the area over which the modulus is $E_1$ be $a_1$, and that over which the modulus is $E_2$ be $a_2$. Then

$$\int_{v=0}^{v=v_1} da = a_1, \text{ and } \int_{v=0}^{v=-v_2} da = a_2.$$

The area of the cross section $a = a_1 + a_2$.

Since the load is applied at the centres of area of the end cross sections, and the column was originally straight, any bending which takes place must be due to the want of uniformity in the modulus of elasticity. Further, on whichever side of the line $CC_0C$ the centre of resistance $D_0$ falls, that side of the column will obviously become the convex side. Let $v = v_1$ and $u = u_1$

be the distances of the extreme convex fibre from the lines $CC_0C$ and $DD_0D$ respectively, and $v = -v_2$ and $u = -u_2$ the corresponding distances to the extreme concave fibre. It then follows from equation (7) that $E_1$, the modulus on the convex side, is greater than $E_2$, the modulus on the concave side. That is to say, the side having the lesser modulus will become the concave side of the column.

Now from equation (2)

$$dW = s_a E \cdot da$$

$$W = s_a \int_{-v_2}^{v_1} E \cdot da$$

$$= s_a \left( E_1 \int_0^{v_1} da + E_2 \int_0^{-v_2} da \right)$$

$$= s_a (E_1 a_1 + E_2 a_2)$$

Hence $$s_a = \frac{W}{E_1 a_1 + E_2 a_2} = \frac{W}{E_a a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

if $E_1 a_1 + E_2 a_2 = E_a a$.

The strain $s_a$ could evidently be found from ordinary direct compression experiments on a short piece of the column. $E_a$ would then be the modulus of elasticity for the short piece as usually determined.

The position of the centre of resistance $D_0$ is fixed by equation (7). Let $v$ and $u$ be the distances of a small tube of fibres $\delta a$ from the lines $CC_0C$ and $DD_0D$ respectively. Then

$$v = u + \epsilon \text{ and } u = v - \epsilon.$$

Hence $$\int_{-u_2}^{u_1} E \cdot u \cdot da = \int_{-v_2}^{v_1} E (v - \epsilon)\, da = \int_{-v_2}^{v_1} Ev \cdot da - \epsilon \int_{-v_2}^{v_1} E \cdot da = 0.$$

Now from $v = 0$ to $v = v_1$, $E = E_1$, and from $v = 0$ to $v = -v_2$, $E = E_2$. Split the integrals

$$E_1 \int_0^{v_1} v \cdot da + E_2 \int_0^{-v_2} v \cdot da - \epsilon E_1 \int_0^{v_1} da - \epsilon E_2 \int_0^{-v_2} da = 0,$$

whence $E_1 a_1 \bar{v}_1 - E_2 a_2 \bar{v}_2 - \epsilon E_1 a_1 - \epsilon E_2 a_2 = 0$, where $\bar{v}_1$ and $\bar{v}_2$ are the distances of the centres of area of the areas $a_1$ and $a_2$ from the line $CC_0C$ respectively.

Hence $$\epsilon = \frac{E_1 a_1 \bar{v}_1 - E_2 a_2 \bar{v}_2}{E_1 a_1 + E_2 a_2}.$$

But since $v = 0$ is the centre of area of the cross section, therefore $a_1 \bar{v}_1 = a_2 \bar{v}_2$. Hence

$$\epsilon = \frac{(E_1 - E_2)\, a_1 \bar{v}_1}{E_1 a_1 + E_2 a_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Now $E_1$ and $E_2$ are supposed to be constant over the entire length of the column. Therefore the value of $\epsilon$ is a constant. That is to say, the line of resistance lies at a constant distance from the central axis. Since W acts along the central axis it is evident that the effect of the want of homogeneity is to produce a virtual eccentricity of loading of value $\epsilon$, and the column is, in effect, an eccentrically loaded column. It may be noted that $\epsilon$ is a function of $\bar{v}_1$ or $\bar{v}_2$, and not of the radius of gyration $\kappa$.

If the cross section be symmetrical with regard to the axis $CC_0C$, $a_1 = a_2 = \frac{a}{2}$.

Then
$$\epsilon = \frac{E_1 - E_2}{E_1 + E_2}\bar{v}_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and generally, if $E_1a_1 + E_2a_2 = E_a a$ and $(E_1 - E_2) = eE_a$, then

$$\epsilon = e\,\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The moment of stiffness S can be obtained from the equation

$$S = \int_{-u_2}^{u_1} E \cdot u^2 \cdot da.$$

Since $u = v - \epsilon$,

$$S = \int_{-v_2}^{v_1} E \cdot v^2 \cdot da - 2\epsilon\int_{-v_2}^{v_1} Ev \cdot da + \epsilon^2\int_{-v_2}^{v_1} E \cdot da,$$

which reduces, by the use of equation (7), to

$$S = \int_{-v_2}^{v_1} Ev^2 \cdot da - \epsilon^2\int_{-v_2}^{v_1} E \cdot da$$

$$= E_1I_1 + E_2I_2 + E_1a_1\bar{v}_1 + E_2a_2\bar{v}_2 - \epsilon^2 (E_1a_1 + E_2a_2).$$

Substituting for $\epsilon$ its value obtained above,

$$S = E_1I_1 + E_2I_2 + \frac{E_1E_2a_1a_2(\bar{v}_1 + \bar{v}_2)^2}{E_1a_1 + E_2a_2} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

where $I_1$ and $I_2$ are the moments of inertia of the areas $a_1$ and $a_2$ about the axes through their centres of area parallel to $CC_0C$.

This expression can be thrown into different shapes by virtue of the relation $a_1\bar{v}_1 = a_2\bar{v}_2$.

If the section be symmetrical about the axis $CC_0C$, $a_1 = a_2 = \frac{a}{2}$,

$$E_1a_1 + E_2a_2 = \left(E_1 + E_2\right)\frac{a}{2} = E_a a.$$

Hence $E_1 + E_2 = 2E_a$. But $E_1 - E_2 = eE_a$. Therefore $E_1E_2 = \left(1 - \frac{e^2}{4}\right)E_a^2$.

Now in all materials used for columns the largest variations in the value of the modulus of elasticity give a value for $e$ not much exceeding $\frac{1}{5}$. That is to say, $E_1E_2 = \left(1 - \frac{1}{100}\right)E_a^2$ or $E_1E_2 = E_a^2$, within 1 per cent. Also $E_1 = \left(1 + \frac{e}{2}\right)E_a$, and $E_2 = \left(1 - \frac{e}{2}\right)E_a$.

Further, $I_1 = I_2$ and $\bar{v}_1 = \bar{v}_2$. Hence

$$S = 2E_aI_1 + \frac{E_a^2a \cdot (2a_1\bar{v}_1^2)}{E_a a} = E_a\left\{2I_1 + 2a_1\bar{v}_1^2\right\} = E_aI.$$

In a symmetrical cross section, therefore, $S = E_aI$ very very nearly.

Similarly in an unsymmetrical cross section, if $E_a^2 = E_1E_2$ and $E_1I_1 + E_2I_2 = E_a(I_1 + I_2)$, which will be approximately correct,

$$S = E_a\left\{I_1 + I_2 + a_1\bar{v}_1^2 + a_2\bar{v}^2\right\} = E_aI \quad . \quad . \quad . \quad . \quad (53)$$

Unless the asymmetry of the cross section be very great, therefore, it may be assumed that $S = E_a I$.

Now it has been shown that the column is, in effect, an eccentrically loaded column, the eccentricity being $\epsilon$ (Fig. 8). The formulæ of Variation 3 will therefore apply. The equation to the bent line of resistance is, therefore, from equation (42),

$$y = \epsilon \sec \frac{aL}{2} \cos ax\,;\ \text{and}\ y_0 = \epsilon \sec \frac{aL}{2},\ \text{where}$$

$$a^2 = \frac{W}{S(1 - s_a)} = \frac{W}{E_a I\left(1 - \dfrac{W}{E_a a}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

$$= \frac{W}{E_a I}\ \text{approximately.}$$

Therefore
$$y_0 = e\,\frac{a_1 \bar{v}_1}{a} \sec \frac{\pi}{2}\sqrt{\frac{W}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

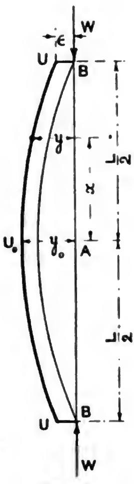


FIG. 8.

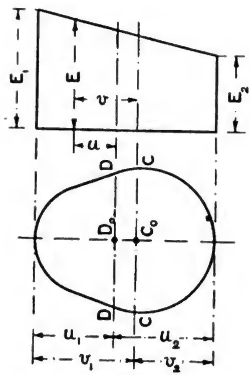


FIG. 9.

The maximum compressive stress which occurs on the concave side at the middle of the column is obtained from equation (18)

$$f_c = -E_2\left\{\frac{W y_0 u_2}{S} + s_1\right\} = E_2\left\{\frac{W}{S}\frac{(v_2 + \epsilon)}{}\sec\frac{aL}{2} + s_a\right\} \quad . \quad (56)$$

(The minus sign is omitted for convenience, but $v_2$ is to be considered as positive.)

Knowing the values of $\epsilon$, $s_a$, and S, the exact value of $f_c$ can be found, or making the same approximations as before,

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left\{1 + \left(v_2 + e\,\frac{a_1\bar{v}_1}{a}\right)\frac{e a_1 \bar{v}_1}{I}\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right\} \quad . \quad . \quad (57)$$

*Assumption* (*b*).—That the modulus of elasticity varies uniformly across the column, but is constant in any longitudinal tube of fibres. Let the modulus where $u = u_1$ be $E_1$, and where $u = u_2$ be $E_2$. As in Assumption (*a*), the centre of resistance will no longer coincide with the centre of area, but the curvature will still be uniplanar, and the two axes, $CC_0C$ and $DD_0D$, perpendicular to the plane of bending. Further, since $D_0$ must fall on the convex side of $CC_0C$, it follows as before that $E_1$ is greater than $E_2$.

Since the modulus of elasticity varies uniformly from one side to the other (see Fig. 9), its value at any point distant $v$ from the axis $CC_0C$, or $u$ from $DD_0D$, is

$$E = E_2 + \frac{E_1 - E_2}{v_1 + v_2}(v + v_2) = \frac{E_1 v_2 + E_2 v_1 + v(E_1 - E_2)}{v_1 + v_2}$$
$$= \frac{E_1 u_2 + E_2 u_1 + u(E_1 - E_2)}{u_1 + u_2}.$$

Now from equation (2), $dW = E \cdot s_a \cdot da$. Hence

$$W = \frac{s_a}{v_1 + v_2}\int_{-v_2}^{v_1}\left\{E_1 v_2 + E_2 v_1 + v(E_1 - E_2)\right\}da$$

$$= \frac{s_a}{v_1 + v_2}\left\{(E_1 v_2 + E_2 v_1)\int_{-v_2}^{v_1} da + (E_1 - E_2)\int_{-v_2}^{v_1} v \cdot da\right\} = \frac{a \cdot s_a}{v_1 + v_2}(E_1 v_2 + E_2 v_1).$$

Therefore $$s_a = \frac{W}{a} \cdot \frac{v_1 + v_2}{E_1 v_2 + E_2 v_1} = \frac{W}{E_a a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where $E_a = \dfrac{E_1 v_2 + E_2 v_1}{v_1 + v_2}$, and is the modulus of elasticity found by direct compression experiments on short pieces of the column.

The position of $D_0$ is determined by equation (7). As before,

$$\int_{-u_2}^{u_1} E \cdot u \cdot da = \int_{-v_2}^{v_1} E(v - \epsilon)\, da = 0.$$

$$\int_{-v_2}^{v_1} Ev \cdot da - \epsilon\int_{-v_2}^{v_1} E \cdot da = 0.$$

Inserting the value of E,

$$\frac{E_1 v_2 + E_2 v_1}{v_1 + v_2}\int_{-v_2}^{v_1} v \cdot da + \frac{E_1 - E_2}{v_1 + v_2}\int_{-v_2}^{v_1} v^2 \cdot da$$
$$- \frac{\epsilon(E_1 v_2 + E_2 v_1)}{v_1 + v_2}\int_{-v_2}^{v_1} da - \frac{\epsilon(E_1 - E_2)}{v_1 + v_2}\int_{-v_2}^{v_1} v \cdot da = 0,$$

whence $$(E_1 - E_2)\,I - \epsilon(E_1 v_2 + E_2 v_1)\,a = 0,$$

or $$\epsilon = \frac{E_1 - E_2}{E_1 v_2 + E_2 v_1}\kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Since $E_1$ and $E_2$ are assumed constant over the length of the column, $\epsilon$ is constant, and the column is in effect eccentrically loaded. If the cross section be symmetrical about the axis $CC_0C$,

$$\epsilon = \frac{E_1 - E_2}{E_1 + E_2} \cdot \frac{2\kappa^2}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

(Compare Neville, 1902); and generally from equation (59), since

$$E_a = \frac{E_1 v_2 + E_2 v_1}{D} \text{ and } E_1 - E_2 = eE_a.$$

$$\epsilon = e\frac{\kappa^2}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The moment of stiffness S can be obtained from the expression

$$S = \int_{-u_2}^{u_1} E \cdot u^2 da = \int_{-v_2}^{v_1} Ev^2 da - \epsilon^2 \int_{-v_2}^{v_1} E \cdot da.$$

Inserting the value of E,

$$S = \frac{E_1 v_2 + E_2 v_1}{v_1 + v_2}\int_{-v_2}^{v_1} v^2 \cdot da + \frac{E_1 - E_2}{v_1 + v_2}\int_{-v_2}^{v_1} v^3 \cdot da$$

$$- \epsilon^2 \frac{E_1 v_2 + E_2 v_1}{v_1 + v_2}\int_{-v_2}^{v_1} da - \epsilon^2 \frac{E_1 - E_2}{v_1 + v_2}\int_{-v_2}^{v_1} v \cdot da$$

$$= \frac{E_1 v_2 + E_2 v_1}{v_1 + v_2} \cdot I + \frac{E_1 - E_2}{v_1 + v_2}\int_{-v_2}^{v_1} v^3 \cdot da - \epsilon^2 a \frac{E_1 v_2 + E_2 v_1}{v_1 + v_2} \quad . \quad . \quad (62)$$

Introducing the values of $\epsilon$ and $E_a$,

$$S = E_a I\left\{1 - \frac{e^2\kappa^2}{D^2}\right\} + E_a \int_{-v_2}^{v_1} v_3 \cdot da \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In this expression, $\frac{e^2\kappa^2}{D^2}$ is obviously a very small fraction. The integral can only be evaluated in special cases. For symmetrical sections its value is zero. Hence in symmetrical sections $S = E_a I$ very nearly, which will also be true for all sections not very asymmetrical. For practical purposes this may be assumed to be the case.

Proceeding as in Assumption (*a*),

$$y_0 = e\frac{\kappa^2}{D} \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and
$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left\{1 + \left(v_2 + e\frac{\kappa^2}{D}\right)\frac{e}{D}\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right\}. \quad . \quad . \quad (65)$$

These are close approximations to the true values.

*Assumption* (*c*).—That the modulus of elasticity varies both in the direction of the width and the length of the column. Under these conditions it is evident that $\epsilon$ will be no longer constant, but will have a different value at different points in the length. In other words, the line of resistance will be no longer straight. In addition, the value of S will vary also. Nevertheless, from what has been proved above it follows that it will be sufficiently exact to assume that S is constant, and equal to $E_a I$. Strictly speaking, $E_a$ is not a constant, but will vary slightly at different points in the length. For practical purposes, however, its mean value may be taken.

The column may be looked upon, therefore, as a column with initial

curvature, to which the analysis of Variation 2 will apply; or more probably, as both initially curved and eccentrically loaded, in which case the formulæ of Variation 6 must be used.

### VARIATION 5. THE NON-HOMOGENEOUS BRACED COLUMN (FIDLER'S ASSUMPTIONS)

The cross section (Fig. 10) consists of two flanges connected together by a stiff web. The effect of secondary flexure in the flanges is supposed to be negligible. The cross section is uniform, and the column originally straight. The load is applied at the centres of area of the end cross sections, and in the direction of the unstrained central axis. The modulus of elasticity is different in each flange, but uniform throughout their length.

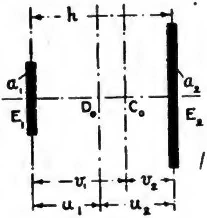


FIG. 10.

Let $E_1$ be the value of the modulus in the convex flange, and $E_2$ that in the concave flange. As in Variation 4, the curvature will be uniplanar, and the axes $CC_0C$ and $DD_0D$ perpendicular to the plane of bending. The centre of resistance $D_0$ will fall on the convex side of $CC_0C$, and $E_1$ will be greater than $E_2$.

Let $a_1$ be the area of the convex flange and $a_2$ that of the concave. Then $a = a_1 + a_2$.

From equation (2) $$s_a = \frac{1}{E} \cdot \frac{dW}{da},$$

whence $$W = s_a \int_{-v_2}^{v_1} E \cdot da = s_a \, (E_1 a_1 + E_2 a_2).$$

Therefore, $$s_a = \frac{W}{E_1 a_1 + E_2 a_2} = \frac{W}{E_a a} \quad \ldots \ldots \quad (66)$$

where $E_a a = E_1 a_1 + E_2 a_2$.

The position of the centre of resistance is determined by equation (7):

$$\int_{-u_2}^{u_1} E u \cdot da = 0,$$

or $$E_1 \int_0^{u_1} u \cdot da + E_2 \int_0^{-u_2} u \cdot da = 0.$$

$$E_1 a_1 u_1 - E_2 u_2 a_2 = 0.$$

Therefore, $$\frac{u_1}{u_2} = \frac{E_2 a_2}{E_1 a_1} \quad \ldots \ldots \quad (67)$$

but $u_1 + u_2 = h$ (Fig. 10). Hence, $$u_1 = \frac{E_2 a_2 h}{E_1 a_1 + E_2 a_2} \quad \ldots \ldots \quad (68)$$

$$u_2 = \frac{E_1 a_1 h}{E_1 a_1 + E_2 a_2} \quad \ldots \ldots \quad (69)$$

Let $C_0D_0$, the distance between the centre of area and the centre of resistance, be $\epsilon$. Then $a_1(u_1 + \epsilon) = a_2(u_2 - \epsilon)$.

$$\epsilon = \frac{a_2u_2 - a_1u_1}{a_1 + a_2} \quad . \quad . \quad . \quad (70)$$

Substituting for $u_1$ and $u_2$, $\epsilon = \dfrac{(E_1 - E_2)\,a_1a_2h}{a\,(E_1a_1 + E_2a_2)}$ . . . (71)

putting as before $(E_1 - E_2) = eE_a$,

$$\epsilon = \frac{ea_1a_2h}{a^2} \quad . \quad . \quad . \quad (72)$$

The value of $\epsilon$ is evidently constant over the whole length of the column, which is in effect an eccentrically loaded column. If the cross section be symmetrical, $a_1 = a_2$ and

$$\epsilon = \frac{E_1 - E_2}{E_1 + E_2} \cdot \frac{h}{2} = \frac{eh}{4} \quad . \quad . \quad . \quad (73)$$

The moment of stiffness S is given by the equation

$$S = \int_{-u_2}^{u_1} Eu^2da = E_1u_1^2a_1 + E_2u_2^2a_2.$$

Introducing the values of $u_1$ and $u_2$ found above,

$$S = \frac{E_1E_2a_1a_2h^2}{E_1a_1 + E_2a_2}. \quad . \quad . \quad . \quad (74)$$

The moments of inertia of the flanges themselves are assumed to be negligible.

Since the column is in effect eccentrically loaded, the formulæ of Variation 3 may be applied. The equation to the bent line of resistance is therefore

$$y = \epsilon \sec\frac{\alpha L}{2} \cos \alpha x \quad . \quad . \quad . \quad (75)$$

and

$$y_0 = \epsilon \sec\frac{\alpha L}{2} \quad . \quad . \quad . \quad (76)$$

where

$$\alpha^2 = \frac{W}{S(1 - s_a)} = \frac{W}{E_aI} \text{ approximately} \quad . \quad . \quad . \quad (77)$$

Therefore

$$y_0 = \frac{ea_1a_2h}{a^2} \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} \quad . \quad . \quad . \quad (78)$$

The maximum compressive stress at the middle of the column on the concave side is given by equation (18):

$$f_c = -E_2\left\{\frac{Wy_0u_2}{S} + s_a\right\}.$$

Substituting the exact values of the various quantities,

$$f_c = E_2\left\{\frac{WE_1a_1h}{E_1a_1 + E_2a_2} \times \frac{(E_1 - E_2)\,a_1a_2h}{a\,(E_1a_1 + E_2a_2)} \times \frac{E_1a_1 + E_2a_2}{E_1E_2a_1a_2h^2} \sec\frac{\alpha L}{2} + \frac{W}{E_1a_1 + E_2a_2}\right\}$$

$$= \frac{W}{E_1a_1 + E_2a_2}\left\{E_2 + \frac{a_1}{a}(E_1 - E_2)\sec\frac{\alpha L}{2}\right\}. \quad . \quad . \quad . \quad (79)$$

or with the usual approximations,

$$f_c = \frac{W}{E_a a}\left\{E_a\left(1 - \frac{e}{2}\right) + \frac{ea_1}{a} E_a \sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right\}$$

$$f_c = f_a\left\{\left(1 - \frac{e}{2}\right) + \frac{ea_1}{a} \sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right\}. \quad (80)$$

In a symmetrical cross section $a_1 = \frac{a}{2}$, and

$$f_c = f_a\left\{1 + \frac{e}{2}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\}. \quad (81)$$

a simple expression for the maximum stress in the column. This expression may be compared with Fidler's equations (1886).

It is worth noticing that since from equation (73) for symmetrical columns, the value of $\epsilon = \frac{eh}{4}$, and as pointed out in the previous Variation the maximum value of $e = \frac{1}{5}$, it follows that the maximum value of $\epsilon = \frac{h}{20}$.

### VARIATION 6. THE ORDINARY COLUMN

In the ordinary column all the different imperfections considered in the preceding Variations will be found. The central axis will have an initial curvature, the load will be eccentric, and the modulus of elasticity will vary both in the direction of the width and of the length of the column. It has been shown, however, that the result of this latter imperfection is in effect an initial curvature of the line of resistance together with an eccentricity of loading. The secondary results of the variation in the modulus, namely, alterations in the values of S and $s_a$, have been shown to be negligible from a practical point of view.

It therefore follows that all such imperfections in a position-fixed column can be taken into account by assuming it to be initially curved and eccentrically loaded.

In general, however, the different imperfections will tend to produce flexure in different planes. Nevertheless, if the fibres on one side of the column have a greater modulus of elasticity than those on the other, the initial curvature and eccentricity due to the variation in the modulus will both cause flexure in the same plane. It is at least possible that the initial curvature of the central axis and the eccentricity of loading may also tend to cause bending in this plane.

As a possible contingency, therefore, it will be assumed that the initial curvature of the central axis, of the line of resistance, and the direction of the eccentricity of loading all lie in the plane perpendicular to the principal axis of elasticity about which the value of S is a minimum. The bending will then be uniplanar.

It may be well to remark that it does not follow that the particular combination of imperfections imagined above is that which, for given values of the load and eccentricities, will produce the greatest stress in the column. It is possible that flexure in two planes at right angles may result in greater stresses. Nevertheless, the direction of these planes cannot be discovered by a simple application of the core theory, and it may be urged that in cases

where experiments have been made in such a manner that the specimen was equally free to deflect in all directions, it actually deflected in the direction corresponding to the least value of $\kappa$ or thereabouts.* Cases are on record † in which specimens, eccentrically loaded so as to deflect in the direction of the greatest value of $\kappa$, failed rather as concentrically loaded specimens in the direction of the least value of $\kappa$.

The column is assumed to be of uniform cross section. Let $VV_1V$ (Fig. 11) be the original shape of the central axis, BB the line of action of the load, and $UU_1U$ the original shape of the line of resistance of the column. Then UB is the eccentricity of the load $= \epsilon_2$. Of this eccentricity, $VB = \epsilon_4$ is due to inaccurate centering, and $UV = \epsilon_6$ to variations in the modulus of elasticity, $\epsilon_2 = \epsilon_4 + \epsilon_6$. $U_1U'$ is the original deflection of the line of resistance $= \epsilon_1$. Of this deflection, $U'U'' = V_1V' = \epsilon_3$ is due to the original deflection of the central axis, and $U_1U'' = \epsilon_5$ is the original deflection due to variations in the modulus of elasticity, $\epsilon_1 = \epsilon_3 + \epsilon_5$.

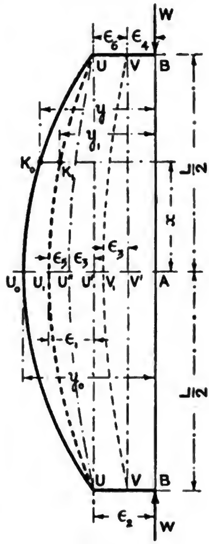

FIG. 11.

It will be assumed that the curve $UU_1U$ is a smooth plane curve, and an arc of a parabola, its exact shape, as has been shown, not being of great importance. Let $UU_0U$ be the final shape taken by the line of resistance. Take origin at A, $AB = \frac{L}{2}$. Let $x$ and $y$ be the co-ordinates of any point $K_0$ on the line of resistance in its final position, and let $K_1$ be the original position of this point and $x$ and $y_1$ its co-ordinates.

The equation to the line $UU_1U$ is

$$y_1 = \epsilon_2 + \epsilon_1\left(1 - \frac{4x^2}{L^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

and
$$\frac{d^2y_1}{dx^2} = -\frac{8\epsilon_1}{L^2}.$$

Hence, from equation (15),

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + \frac{Wy}{S(1 - s_a)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

It will be assumed, on the grounds before advanced, that S and $s_a$ are constant. Let $a^2 = \frac{W}{S(1 - s_a)}$.

Then
$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + a^2y = 0,$$

to which the solution is

$$y = m \sin ax + n \cos ax - \frac{8\epsilon_1}{a^2L^2},$$

* Christie, 1884. † Tetmajer, 1890.

when $x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$; when $x = \frac{L}{2}$, $y = \epsilon_2$, and

$$n = \left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}.$$

Hence
$$y = \left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}\cos ax - \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad . \quad (84)$$

and the maximum deflection

$$y_0 = \left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2} = \epsilon_2 \sec\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\left(\sec\frac{aL}{2} - 1\right) \quad (85)$$

$y_0$ is evidently the sum of the two deflections consequent upon $\epsilon_1$ and $\epsilon_2$ separately considered.

If the same approximations be made as in the previous cases,

$$y_0 = \epsilon_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \frac{8P\epsilon_1}{\pi^2W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right). \quad . \quad . \quad . \quad (86)$$

The maximum compressive stress, which occurs on the concave side of the column at its centre, is obtained from equation (18):

$$f_c = -E_2\left\{\frac{Wy_0u_2}{S} + s_a\right\}.$$

Now $u_2 = v_2 + \epsilon_5 + \epsilon_6$. Hence, inserting the value of $y_0$, and neglecting the minus sign,

$$f_c = E_2\left[(v_2 + \epsilon_5 + \epsilon_6)\frac{W}{S}\left\{\epsilon_2 \sec\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\left(\sec\frac{aL}{2} - 1\right)\right\} + s_a\right] \quad (87)$$

If now, as in Variation 4, Assumption (*a*), it be supposed that in any cross section the modulus of elasticity on the concave side of the column to the right of the principal axis $CC_0C$ (Fig. 7) is constant, and that the modulus on the convex side, to the left of that axis, is also constant, but different in value, then the axis $CC_0C$ will be perpendicular to the plane of flexure, and it may be shown as before that, if $E_1$ and $E_2$ be the greatest and least values of the modulus at the central cross section, then

$$\epsilon_5 + \epsilon_6 = e \cdot \frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where $e = \frac{E_1 - E_2}{E_a}$, the fractional variation of the modulus of elasticity.

Similarly, $\epsilon_5 = e_5\frac{a_1\bar{v}_1}{a}$ and $\epsilon_6 = e_6\frac{a_1\bar{v}_1}{a}$ where $e_5$ and $e_6$ are the fractional variations of the modulus corresponding to $\epsilon_5$ and $\epsilon_6$. Then $e = e_5 + e_6$ and

$$\epsilon_1 = \epsilon_3 + \epsilon_5 = \epsilon_3 + e_5\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

$$\epsilon_2 = \epsilon_4 + \epsilon_6 = \epsilon_4 + e_6\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Making the same approximations as in previous cases:

$$f_c = E_a\left(1 - \frac{e}{2}\right)\left[\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{W}{E_a I}\left\{\epsilon_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \epsilon_1\frac{8P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\} + \frac{W}{E_a a}\right]$$

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa^2}\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \epsilon_1\frac{8P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\}\right] \quad (91)$$

Expand the secants

$$\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} = 1 + \frac{1}{2}\cdot\frac{\pi^2}{4}\cdot\frac{W}{P} + \frac{5}{24}\cdot\frac{\pi^4}{16}\cdot\frac{W^2}{P^2} + \ldots$$

$$\frac{8P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right) = \frac{8P}{\pi^2 W}\left(\frac{1}{2}\cdot\frac{\pi^2}{4}\cdot\frac{W}{P} + \frac{5}{24}\cdot\frac{\pi^4}{16}\cdot\frac{W^2}{P^2} + \ldots\right)$$

$$= 1 + \frac{5}{12}\cdot\frac{\pi^2}{4}\cdot\frac{W}{P} + \ldots$$

Now in no practical column is the factor of safety for Euler's formula likely to be less than 4 or 5, that is to say, $\frac{W}{P}$ will never be greater than $\frac{1}{5}$ or $\frac{1}{4}$. Hence higher powers of $\frac{W}{P}$ than the first may be neglected. The expression for $f_c$ then becomes

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa^2}\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2\left(1 + \frac{\pi^2}{8}\cdot\frac{W}{P}\right) + \epsilon_1\left(1 + \frac{5\pi^2}{48}\cdot\frac{W}{P}\right)\right\}\right] \quad (92)$$

An even simpler approximation may be obtained by use of the table and curve given on pp. 29 and 30.

From these it will be seen that for all practical cases, that is to say, for those in which the ratio $\frac{W}{P}$ varies from 0 to $\frac{1}{4}$, the values of the two functions

$$\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} \text{ and } \frac{8P}{\pi^2 W}\left\{\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right\}$$

are very nearly equal, and further, that they differ very little from the straight line $\left(1 + \frac{3}{2}\cdot\frac{W}{P}\right)$. For example, if $\frac{W}{P} = \frac{1}{5}$

$$\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} = 1{\cdot}3102$$

$$1 + \frac{3}{2}\cdot\frac{W}{P} = 1{\cdot}30$$

$$\frac{8P}{\pi^2 W}\left\{\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right\} = 1{\cdot}2572.$$

It is evident that within the limits named, which include all properly designed practical cases, for a factor of safety of at least four would always

be used with Euler's formula,* the trigonometrical functions can with perfect safety be replaced by the single algebraic expression (see the straight line on Fig. 5). For working loads and stresses, therefore, equation (91) becomes

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa^2}\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\left\{(\epsilon_1 + \epsilon_2)\left(1 + \frac{3W}{2P}\right)\right\}\right] \quad . \quad (93)$$

For practical use it is possible to simplify this further. It has been pointed out that the maximum value of $e$ is in the neighbourhood of $\frac{1}{5}$. The ratio $\frac{a_1}{a}$ will not differ much from $\frac{1}{2}$, and $\bar{v}_1$ will, in general, be less than $v_2$. The term $\frac{a_1\bar{v}_1}{a}$ will therefore not be large compared with $v_2$, and no great error will be introduced if the two factors containing $e$ be neglected, for both errors are small, and they tend to neutralize one another. In this case equation (93) becomes

$$f_c = \frac{W}{a}\left[1 + \left(1 + \frac{3W}{2P}\right)\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right]. \quad . \quad . \quad . \quad . \quad . \quad (93\text{A})$$

Further, since the ratio of $\frac{W}{P}$ is not likely to be greater than $\frac{1}{5}$, and in the vast majority of columns very much smaller, the factor $\left(1 + \frac{3W}{2P}\right)$ may be replaced by its superior limit 1·3. The expression for $f_c$ then becomes

$$f_c = \frac{W}{a}\left\{1 + 1\cdot 3\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\} = \frac{W}{a}\left\{1 + 1\cdot 3\frac{\epsilon_1 + \epsilon_2}{\omega_2}\right\} \quad . \quad . \quad (94)$$

The maximum tensile stress, which occurs on the convex side of the column at its centre, is obtained from equation (17)

$$f_t = E_1\left\{\frac{Wy_0u_1}{S} - s_a\right\}.$$

Now $u_1 = v_1 - \epsilon_5 - \epsilon_6$,

$$f_t = E_1\left[(v_1 - \epsilon_5 - \epsilon_6)\frac{W}{S}\left\{\epsilon_2\sec\frac{\alpha L}{2} + \frac{8\epsilon_1}{\alpha^2L^2}\left(\sec\frac{\alpha L}{2} - 1\right)\right\} - s_a\right] \quad (95)$$

which reduces, as before, to

$$f_t = E_a\left(1 + \frac{e}{2}\right)\left[\left(v_1 - e\frac{a_1\bar{v}_1}{a}\right)\frac{W}{E_aI}\left\{\epsilon_2\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \epsilon_1\frac{8P}{\pi^2W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\} - \frac{W}{E_aa}\right]$$

$$f_t = \frac{W}{a}\left(1 + \frac{e}{2}\right)\left[\frac{1}{\kappa_2}\left(v_1 - e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2\sec\frac{\pi}{2}\sqrt{\frac{W}{P_1}} + \epsilon_1\frac{8P}{\pi^2W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\} - 1\right] \quad (96)$$

* In the vast majority of practical cases, the ratio $\frac{W}{P}$ is, of course, very much smaller than $\frac{1}{4}$.

Expanding the secants,

$$f_t=\frac{W}{a}\left(1+\frac{e}{2}\right)\left[\frac{1}{\kappa^2}\left(v_1-e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2\left(1+\frac{\pi^2}{8}\cdot\frac{W}{P}\right)+\epsilon_1\left(1+\frac{5\pi^2}{48}\cdot\frac{W}{P}\right)\right\}-1\right] \quad (97)$$

or alternatively,

$$f_t=\frac{W}{a}\left(1+\frac{e}{2}\right)\left[\frac{1}{\kappa^2}\left(v_1-e\frac{a_1\bar{v}_1}{a}\right)\left\{(\epsilon_1+\epsilon_2)\left(1+\frac{3W}{2P}\right)\right\}-1\right] \quad . \quad (98)$$

which becomes approximately

$$f_t=\frac{W}{a}\left[1\cdot 3\,\frac{v_1}{\kappa^2}(\epsilon_1+\epsilon_2)-1\right]=\frac{W}{a}\left[1\cdot 3\,\frac{\epsilon_1+\epsilon_2}{\omega_1}-1\right] \quad . \quad . \quad . \quad (99)$$

From equation (97) onwards these formulæ only apply to cases in which $\frac{W}{P}<\frac{1}{4}$. The final approximation, in which the two factors containing $e$ are neglected, is not so satisfactory as in the equation for $f_c$; for although the two errors introduced tend to neutralize one another, the value of $f_t$ is reduced by neglecting $\left(1+\frac{e}{2}\right)$, whereas the value of $f_c$ is increased by neglecting $\left(1-\frac{e}{2}\right)$.

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Returning to equation (87), another approximation can be obtained as follows:

$$f_c=E_2\left[(v_2+\epsilon_5+\epsilon_6)\frac{W}{S}\cdot\frac{8}{\alpha^2L^2}\left\{\epsilon_2\frac{\alpha^2L^2}{8}\sec\frac{\alpha L}{2}+\epsilon_1\left(\sec\frac{\alpha L}{2}-1\right)\right\}+s_a\right].$$

But $\alpha^2=\frac{W}{S(1-s_a)}$, hence

$$f_c=E_2\left[(v_2+\epsilon_5+\epsilon_6)\frac{8(1-s_a)}{L^2}\left\{\epsilon_2\frac{\alpha^2L^2}{8}\sec\frac{\alpha L}{2}+\epsilon_1\left(\sec\frac{\alpha L}{2}-1\right)\right\}+s_a\right].$$

Now $s_a$ may be neglected in comparison with unity, $\frac{\alpha L}{2}=\frac{\pi}{2}\sqrt{\frac{W}{P}}$ very nearly, and $s_a=\frac{W}{E_a a}$. The equation becomes, therefore,

$$f_c=E_a\left(1-\frac{e}{2}\right)\left[\left(v_2+e\frac{a_1\bar{v}_1}{a}\right)\cdot\frac{8}{L^2}\left\{\epsilon_2\cdot\frac{\pi^2W}{8P}\cdot\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right.\right.$$

$$\left.\left.+\epsilon_1\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}-1\right)\right\}+\frac{W}{E_a a}\right] \quad (100)$$

Now it has been shown above that, for cases in which $\frac{W}{P}<\frac{1}{4}$,

$$\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}=1+\frac{3}{2}\cdot\frac{W}{P}=\frac{8P}{\pi^2W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}-1\right).$$

Hence $$\frac{\pi^2W}{8P}\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}=\frac{\pi^2W}{8P}\left(1+\frac{3}{2}\cdot\frac{W}{P}\right)=\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}-1\right).$$

Replacing the trigonometrical functions by the algebraic expression,

$$f_c=E_a\left(1-\frac{e}{2}\right)\left[\left\{\left(v_2+e\frac{a_1v_1}{a}\right)(\epsilon_1+\epsilon_2)\frac{\pi^2W}{L^2P}\left(1+\frac{3}{2}\cdot\frac{W}{P}\right)\right\}+\frac{W}{E_a a}\right] (101)$$

Neglecting the two terms containing $e$, and inserting the value $\frac{1}{5}$ for $\frac{W}{P}$, this expression becomes

$$f_c = \frac{W}{a} + 2 \cdot 6 \frac{E_a v_2}{L^2} (\epsilon_1 + \epsilon_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

which might have been obtained directly from equation (94).

## *COLUMNS WITH FIXING MOMENTS AT THEIR ENDS*

Let $UU_0U$, Fig. 12, represent the line of resistance of a column with fixing moments at both its ends, and $K_0$ be any point thereon. Let A be the origin, and $x$ and $y$ the co-ordinates of $K_0$. Let $M_a$ be the fixing moment at the lower end of the column, and $M_b$ that at the upper.

Then if $M_a$ be not equal to $M_b$, there is an unbalanced moment $M_a - M_b$ tending to overturn the column. This will call into play equal horizontal forces FF at each end of the column, such that $FL = M_a - M_b$.

The bending moment at $K_0$, due to the vertical load W, is $Wy$. That due to the lower horizontal force F is $Fx = (M_a - M_b)\frac{x}{L}$. Hence the total moment at the point $K_0$ is $M = Wy + M_a - (M_a - M_b)\frac{x}{L}$.

Equation (15), giving the shape of the bent line of resistance of the column, becomes, therefore,

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a - (M_a - M_b)\frac{x}{L}}{S(1 - s_a)} = 0 \quad . \quad . \quad . \quad (103)$$

The stress anywhere, in terms of the bending moment, from equation (16) is

$$f = E\left\{\frac{Mu}{S} - s_a\right\} = E\left[\frac{u}{S}\left\{Wy + M_a - (M_a - M_b)\frac{x}{L}\right\} - s_a\right]. \quad (104)$$

In cases where $M_a = M_b$, equation (103) becomes

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a}{S(1 - s_a)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105$$

Equation (104) becomes

$$f = E\left[\frac{u}{S}\left\{Wy + M_a\right\} - s_a\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

and $F = 0$. It is more convenient, under these circumstances, to take origin at the centre of the load line.

## CASE II. Position- and Direction-fixed Columns. Uniplanar Bending

*Both ends fixed in position and direction*

### Variation I. Ideal Conditions

The column is of uniform cross section and originally perfectly straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross

sections and in the direction of the unstrained central axis. Suppose the column to bend.

Since the modulus of elasticity is constant, it follows from equation (2) that $s_a = \dfrac{W}{Ea}$.

The centre of resistance will coincide with the centre of area of the cross section, and the central axis will be the line of resistance. The moment of stiffness S will be constant, and equal to EI, where I is the least moment of

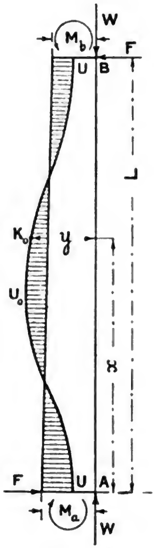



FIG. 12.

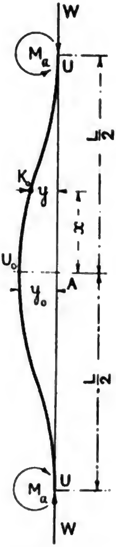



FIG. 13.

inertia of the cross section. Since the column was originally straight, the initial curvature $\dfrac{1}{\rho_1}$ will be zero.

Let $UU_0U$, Fig. 13, be the shape of the bent line of resistance (the central axis of the column). Take origin at A in the line of action of W, $AU = \dfrac{L}{2}$.
Let $x$ and $y$ be the co-ordinates of any point $K_0$ in the line of resistance.

From the symmetry of the figure the fixing moments at each end are equal, and $M_a = M_b$; hence, from equation (105),

$$\frac{d^2y}{dx^2} + \frac{Wy + M_a}{EI\left(1 - \dfrac{W}{Ea}\right)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Let
$$\frac{W}{I\left(E - \frac{W}{a}\right)} = \alpha^2$$

Then
$$\frac{d^2y}{dx^2} + \alpha^2 y + \frac{M_a \alpha^2}{W} = 0,$$

to which the solution is $y + \frac{M_a}{W} = m \sin \alpha x + n \cos \alpha x$.

Now when
$$x = 0, \frac{dy}{dx} = 0 \text{ and } m = 0.$$

Hence,
$$y + \frac{M_a}{W} = n \cos \alpha x.$$

When $x = 0$, $y = y_0$; therefore $n = y_0 + \frac{M_a}{W}$, and the equation to the bent line of resistance is

$$y = \left(y_0 + \frac{M_a}{W}\right) \cos \alpha x - \frac{M_a}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

But when $x = \pm \frac{L}{2}, \frac{dy}{dx} = 0$; that is, $0 = -\left(y_0 + \frac{M_a}{W}\right) \alpha \sin \frac{\alpha L}{2}$.

Hence, either $\left(y_0 + \frac{M_a}{W}\right) = 0$, or $\sin \frac{\alpha L}{2} = 0$. But if $\left(y_0 + \frac{M_a}{W}\right) = 0, \frac{dy}{dx} = 0$, whatever be the value of $x$, and the column will remain straight. Therefore, if the column bend, $\sin \frac{\alpha L}{2} = 0$ and $\frac{\alpha L}{2} = r\pi$. For the practical case $r = 1$,

$$\alpha^2 = \frac{4\pi^2}{L^2} = \frac{W}{I\left(E - \frac{W}{a}\right)}$$

whence
$$W = \frac{\frac{4\pi^2 EI}{L^2}}{1 + \frac{4\pi^2 I}{aL^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

If $\frac{W}{a}$ be neglected in comparison with E,

$$W = \frac{4\pi^2 EI}{L^2} = P_2,$$

the usual Eulerian value.

### Variation 2. Column with Initial Curvature

The column is of uniform cross section, but not originally straight. The modulus of elasticity is constant everywhere and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and along the line joining its points of application. The ends are held fixed in their original position and direction. The effect of the load will be to increase the central deflection.

Then, as in Variation 1, $s_a = \frac{W}{Ea}$, $S = EI$ = constant, the central axis will be the line of resistance, and I the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending.

Let $UU_1U$ and $UU_0U$, Fig. 14, be the initial and final positions of the line of resistance. Take origin at A. $AU = \frac{L}{2}$. Let the co-ordinates of any point $K_0$ in the line of resistance in its final position be $x$ and $y$. Let $K_1$ be the initial position of the point $K_0$, and $x$ and $y_1$ its co-ordinates.

Since by symmetry the fixing moments $M_a$ and $M_b$ are equal, from equation (105)

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a}{EI\left(1 - \frac{W}{Ea}\right)} = 0 \quad . \quad . \quad (110)$$

Assume that the original shape of the line of resistance was a parabolic arc to which the equation is

$$y_1 = \epsilon_1\left\{1 - \frac{4x^2}{L^2}\right\} \quad . \quad . \quad . \quad . \quad (111)$$

Then $\frac{dy_1}{dx} = -\frac{8\epsilon_1 x}{L^2}$ and $\frac{d^2y_1}{dx^2} = -\frac{8\epsilon_1}{L^2}$

when $x = \frac{L}{2}$, $\frac{dy_1}{dx} = -\frac{4\epsilon_1}{L}$.

Equation (110) becomes, therefore,

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + \frac{Wy + M_a}{EI\left(1 - \frac{W}{Ea}\right)} = 0 \quad . \quad . \quad (112)$$

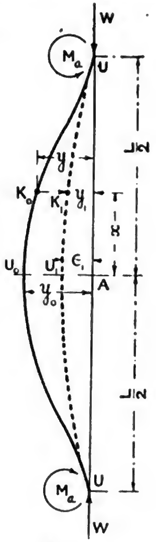



FIG. 14

Let

$$a^2 = \frac{W}{I\left(E - \frac{W}{a}\right)}, \text{ then } \frac{d^2y}{dx^2} + a^2y + \frac{M_a a^2}{W} + \frac{8\epsilon_1}{L^2} = 0,$$

to which the solution is

$$y + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = m \sin ax + n \cos ax.$$

When $x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$.

When $x = 0$, $y = y_0$, therefore $n = y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}$,

and the equation to the line of resistance in its final position is

$$y + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}\right)\cos ax \quad . \quad . \quad . \quad (113)$$

Since the ends of the column are fixed, both in position and direction, when $x = \frac{L}{2}$, $y = 0$,

and
$$\frac{dy}{dx} = \frac{dy_1}{dx} = -\frac{4\epsilon_1}{L} \qquad (114)$$

Hence,
$$\frac{8\epsilon_1}{\alpha^2 L^2} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{\alpha^2 L^2} + \frac{M_a}{W}\right)\cos\frac{\alpha L}{2}$$

and
$$\frac{4\epsilon_1}{L} = \alpha\left(y_0 + \frac{8\epsilon_1}{\alpha^2 L^2} + \frac{M_a}{W}\right)\sin\frac{\alpha L}{2} \qquad (115)$$

from which
$$\frac{M_a}{W} = -\frac{8\epsilon_1}{\alpha^2 L^2} + \frac{4\epsilon_1}{\alpha L}\cot\frac{\alpha L}{2}$$

and
$$M_a = -\frac{8\epsilon_1 W}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right) \qquad (116)$$

Now from equation (115),
$$y_0 = \frac{4\epsilon_1}{\alpha L \sin\frac{\alpha L}{2}} - \frac{8\epsilon_1}{\alpha^2 L^2} - \frac{M_a}{W} = \frac{4\epsilon_1}{\alpha L \sin\frac{\alpha L}{2}} - \frac{4\epsilon_1}{\alpha L}\cot\frac{\alpha L}{2}.$$

Hence,
$$y_0 = \frac{4\epsilon_1}{\alpha L}\left\{\operatorname{cosec}\frac{\alpha L}{2} - \cot\frac{\alpha L}{2}\right\} \qquad (117)$$

$$= \frac{4\epsilon_1}{\alpha L}\tan\frac{\alpha L}{4} \qquad (118)$$

and from equations (113) and (115)
$$y = \frac{4\epsilon_1}{\alpha L \sin\frac{\alpha L}{2}}\left\{\cos\alpha x - \cos\frac{\alpha L}{2}\right\} \qquad (119)$$

The maximum compressive stress in the column may occur either at the centre or the ends. From equation (18) the maximum compressive stress at the centre is
$$f_c = -E_2\left\{\frac{Mu_2}{S} + s_a\right\}.$$

In the present case, neglecting the negative sign of compression,
$$f_c = E\left\{v_2\frac{Wy_0 + M_a}{EI} + \frac{W}{Ea}\right\}$$
$$= \frac{Wv_2}{I}\left(y_0 + \frac{M_a}{W}\right) + \frac{W}{a}$$
$$= \frac{Wv_2}{I}\left(\frac{4\epsilon_1}{\alpha L \sin\frac{\alpha L}{2}} - \frac{8\epsilon_1}{\alpha^2 L^2}\right) + \frac{W}{a}$$
$$= \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_2}{\kappa^2}\cdot\frac{1}{\alpha^2 L^2}\left(\frac{\alpha L}{2}\operatorname{cosec}\frac{\alpha L}{2} - 1\right)\right\} \qquad (120)$$

Now $\alpha^2 = \frac{W}{I\left(E - \frac{W}{a}\right)} = \frac{W}{EI}$, if $\frac{W}{a}$ be neglected in comparison with E.

Hence $\alpha^2L^2 = 4\pi^2 \cdot \frac{W}{P_2}$, and $\frac{\alpha L}{2} = \pi\sqrt{\frac{W}{P_2}}$ where $P_2 = \frac{4\pi^2EI}{L^2}$, Euler's crippling load for a position- and direction-fixed column.

Making these substitutions,

$$f_c = \frac{W}{a}\left\{1 + \frac{2\epsilon_1 v_2}{\kappa^2} \cdot \frac{P_2}{\pi^2 W}\left(\pi\sqrt{\frac{W}{P_2}} \operatorname{cosec} \pi\sqrt{\frac{W}{P_2}} - 1\right)\right\} \quad . \quad . \quad (121)$$

an expression for the maximum compressive stress at the centre of the column.

At the ends of the column the bending moment is $M_a$. Equation (106), for the maximum compressive stress there, becomes,

$$f_c = E\left\{\frac{M_a u}{EI} - \frac{W}{Ea}\right\}$$

The maximum compressive stress will occur where $u = v_1$, hence

$$f_c = -\left\{-v_1\frac{M_a}{I} + \frac{W}{a}\right\}$$

Neglecting the negative sign of compression,

$$f_c = \frac{W}{a} + \frac{v_1}{I} \cdot \frac{8\epsilon_1 W}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)$$

$$= \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_1}{\kappa^2} \cdot \frac{1}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)\right\} \quad . \quad . \quad . \quad (122)$$

Making the same approximations as before,

$$f_c = \frac{W}{a}\left\{1 + \frac{2\epsilon_1 v_1}{\kappa^2} \cdot \frac{P_2}{\pi^2 W}\left(1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right)\right\} \quad . \quad . \quad (123)$$

an equation giving the maximum compressive stress at the ends.

Comparing the expressions for the maximum compressive stress at the centre and ends, equations (120) and (122):

At the centre, $f_c = \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_2}{\kappa^2} \cdot \frac{1}{\alpha^2 L^2}\left(\frac{\alpha L}{2}\operatorname{cosec}\frac{\alpha L}{2} - 1\right)\right\}$

At the ends, $f_c = \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_1}{\kappa^2} \cdot \frac{1}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)\right\}$

These expressions differ only in the trigonometrical functions representing $f_b$, the stress due to bending,

$$\frac{f_b\ (\text{centre})}{f_b\ (\text{ends})} = \frac{v_2\left(\frac{\alpha L}{2}\operatorname{cosec}\frac{\alpha L}{2} - 1\right)}{v_1\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)} \quad . \quad . \quad . \quad . \quad . \quad (124)$$

Now $\frac{\alpha L}{2} = \pi\sqrt{\frac{W}{P_2}}$ very very nearly, and in practical columns, assuming that the working load will never be greater than $\frac{1}{4}$ of Euler's crippling load, $\frac{W}{P_2}$ varies from 0 to $\frac{1}{4}$.

Hence for practical columns, $\frac{\alpha L}{2}$ varies from 0 to $\frac{\pi}{2}$. But the limiting

value of the function in equation (124), when $\frac{\alpha L}{2} = 0$, is $\frac{1}{2}$, and when $\frac{\alpha L}{2} = \frac{\pi}{2}$, is 0·5708; or, for practical purposes, over this range

$$\frac{f_b\ (\text{centre})}{f_b\ (\text{ends})} = \frac{v_2}{2v_1}$$

In symmetrical cross sections, therefore, where $v_1 = v_2$, the stress due to bending is twice as great at the ends as at the centre; and the maximum compressive stress in any column will be greater at the ends than at the centre unless $v_2$ is double $v_1$. This result, which is analogous to that for direction-fixed beams, is only true if $\frac{W}{P_2} < \frac{1}{4}$.

When W, the actual load on a column, approaches $P_2$, Euler's crippling load, which might be the case in a long slender column at the moment of crippling, $\frac{\alpha L}{2}$ would approach in value to $\pi$, and in the limit when $\frac{\alpha L}{2} = \pi$, the value of the function in equation (124) is unity; hence,

$$\frac{f_b\ (\text{centre})}{f_b\ (\text{ends})} = \frac{v_2}{v_1}.$$

In symmetrical cross sections, therefore, the stress due to bending, and hence the total stress, would be equal at the centre and the ends; and in unsymmetrical sections would be greater at the centre or the ends depending on whether $v_2$ or $v_1$ be the greater.

It is evident, therefore, that under working loads the stress might be greater at the ends, whilst at the failure point it might be greater at the centre.

---

Had the original shape of the line of resistance been sinusoidal instead of a parabolic arc, the following results would have been reached.

$$y_1 = \epsilon_1 \cos \frac{\pi}{L} x \quad \ldots \quad (125)$$

$$M_a = \frac{W\pi\epsilon_1 \alpha L}{\alpha^2 L^2 - \pi^2} \cot \frac{\alpha L}{2} \quad \ldots \quad (126)$$

$$y = \frac{\pi\epsilon_1}{\alpha^2 L^2 - \pi^2}\left\{\alpha L \frac{\cos \alpha x - \cos \frac{\alpha L}{2}}{\sin \frac{\alpha L}{2}} - \pi \cos \frac{\pi}{L} x\right\} \quad \ldots \quad (127)$$

$$y_0 = \frac{\pi\epsilon_1}{\alpha^2 L^2 - \pi^2}\left\{\alpha L \frac{1 - \cos \frac{\alpha L}{2}}{\sin \frac{\alpha L}{2}} - \pi\right\} \quad \ldots \quad (128)$$

$$= \frac{\pi\epsilon_1}{\alpha^2 L^2 - \pi^2}\left\{\alpha L \tan \frac{\alpha L}{4} - \pi\right\} \quad \ldots \quad (129)$$

The maximum compressive stress at the centre is

$$f_c = \frac{W}{a}\left[1 + \frac{2\epsilon_1 v_2}{\kappa^2}\cdot\frac{\pi}{a^2L^2 - \pi^2}\left\{\frac{aL}{2}\operatorname{cosec}\frac{aL}{2} - \frac{\pi}{2}\right\}\right]$$

$$= \frac{W}{a}\left[1 + \frac{2\epsilon_1 v_2}{\pi\kappa^2}\cdot\frac{P_2}{P_2 - 4W}\left\{\frac{\pi}{2} - \pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}\right\}\right] \quad . \quad (130)$$

and at the ends,

$$f_c = \frac{W}{a}\left[1 - \frac{2\epsilon_1 v_1}{\kappa^2}\cdot\frac{\pi}{a^2L^2 - \pi^2}\left\{\frac{aL}{2}\cot\frac{aL}{2}\right\}\right]$$

$$= \frac{W}{a}\left[1 + \frac{2\epsilon_1 v_1}{\pi\kappa^2}\cdot\frac{P_2}{P_2 - 4W}\cdot\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right] \quad . \quad . \quad . \quad . \quad (131)$$

These values do not differ appreciably from those for the parabolic arc.

---

From equation (121) the value of $f_b$, the stress due to bending at the centre of the column, is

$$f_b = \frac{W}{a}\cdot\frac{2\epsilon_1 v_2}{\kappa^2}\,\frac{P_2}{\pi^2 W}\left(\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right),$$

from which

$$\frac{f_b a}{W}\cdot\frac{\kappa^2}{2\epsilon_1 v_2} = \frac{P_2}{\pi^2 W}\left(\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right).$$

Similarly, from equation (123) for the stress at the ends of the column,

$$\frac{f_b a}{W}\cdot\frac{\kappa^2}{2\epsilon_1 v_1} = \frac{P_2}{\pi^2 W}\left(1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right).$$

If the initial curvature be sinusoidal, from equation (130) for the stress at the centre of the column,

$$\frac{f_b a}{W}\cdot\frac{\kappa^2}{2\epsilon_1 v_2} = \frac{1}{\pi}\,\frac{P_2}{P_2 - 4W}\left\{\frac{\pi}{2} - \pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}\right\}$$

and from equation (131) for the stress at the ends of the column,

$$\frac{f_b a}{W}\cdot\frac{\kappa^2}{2\epsilon_1 v_1} = \frac{1}{\pi}\,\frac{P_2}{P_2 - 4W}\,\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}$$

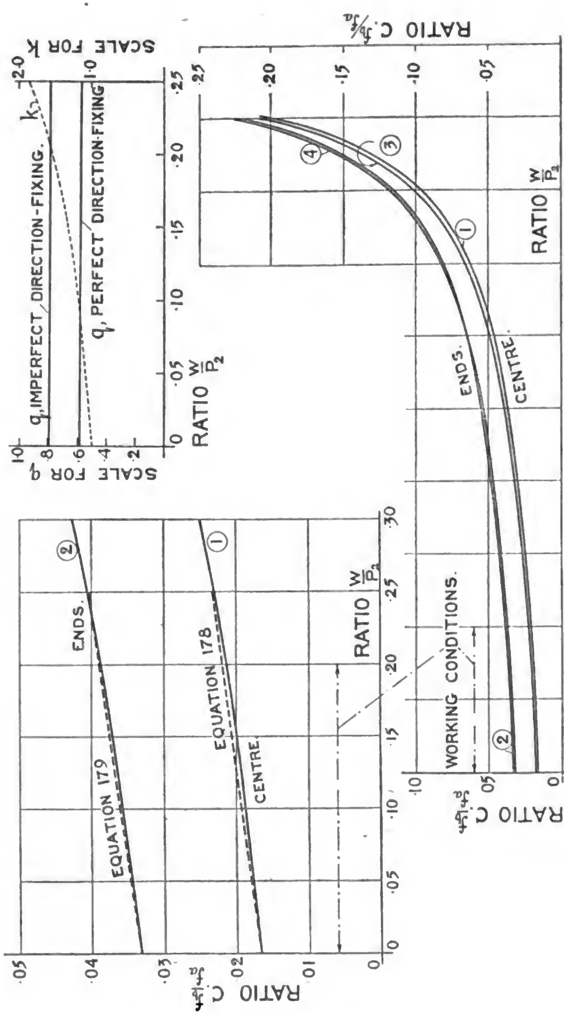
Now, if $\quad \dfrac{\kappa^2}{2\epsilon_1 v_1} = C_1$, and $\dfrac{\kappa^2}{2\epsilon_1 v_2} = C_2$,

where $C_1$ and $C_2$ are constants for any one column, then:

*Parabolic Initial Curvature*

At centre, $C_2\dfrac{f_b}{f_a} = \dfrac{P_2}{\pi^2 W}\left(\pi\sqrt{\dfrac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\dfrac{W}{P_2}} - 1\right)$ . . . . curve 1

At ends, $C_1\dfrac{f_b}{f_a} = \dfrac{P_2}{\pi^2 W}\left(1 - \pi\sqrt{\dfrac{W}{P_2}}\cot\pi\sqrt{\dfrac{W}{P_2}}\right)$ . . . . . . curve 2



RATIO C
RATIO W/P2
EQUATION 179
EQUATION 178
ENDS.
CENTRE.
WORKING CONDITIONS.
SCALE FOR q
SCALE FOR k
q, IMPERFECT DIRECTION-FIXING.
q, PERFECT DIRECTION-FIXING.


Fig. 15.

*Sinusoidal Initial Curvature*

At centre, $C_2 \frac{f_b}{f_a} = \frac{1}{\pi} \frac{P_2}{P_2 - 4W} \left( \frac{\pi}{2} - \pi \sqrt{\frac{W}{P_2}} \operatorname{cosec} \pi \sqrt{\frac{W}{P_2}} \right)$ . . . curve 3

At ends, $C_1 \frac{f_b}{f_a} = \frac{1}{\pi} \frac{P_2}{P_2 - 4W} \pi \sqrt{\frac{W}{P_2}} \cot \pi \sqrt{\frac{W}{P_2}}$ . . . . . . curve 4

These values of $C_1 \frac{f_b}{f_a}$ and $C_2 \frac{f_b}{f_a}$ are plotted on Fig. 15, the curves being numbered to correspond. They show clearly that it makes little difference whether the initial curvature be parabolic or sinusoidal; and if $v_1 = v_2$, and $C_1 = C_2$, that whilst at small loads the stress due to bending at ends is double that of the centre, when the load approaches Euler's limit the stress at the centre approaches more and more in value to that at the ends. The shape of these curves under working conditions $\left( \frac{W}{P_2} < \frac{1}{5} \right)$ should be noted. The calculated values of the functions will be found in the table on p. 56.

## VARIATION 3. THE ECCENTRICALLY LOADED COLUMN

The column is of uniform cross section, and originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at a distance $\epsilon_2$ from the centre of area of the cross section at each end, the points of application lying on the same principal axis and on the same side of the central axis at each end. The ends are held fixed in their original position and direction.

Suppose the column to bend. Then, as in Variation 1, $s_a = \frac{W}{Ea}$, $S = EI = \text{const.}$, the central axis will be the line of resistance, and I is the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending. The initial curvature $\frac{1}{\rho_1}$ will be zero.

Let $UU_0U$ be the shape of the line of resistance of the bent column, Fig. 16, and BB the points of application of the load. $UB = \epsilon_2$. Take origin at A. $AB = \frac{L}{2}$. Let the co-ordinates of any point $K_0$ on the line of resistance be $x$ and $y$.

Since by symmetry $M_a = M_b$, and the column was originally straight, by equation (105)

$$\frac{d^2y}{dx^2} + \frac{Wy + M_a}{EI\left(1 - \frac{W}{Ea}\right)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (132)$$

Let
$$\frac{W}{I\left(E - \frac{W}{a}\right)} = a^2.$$

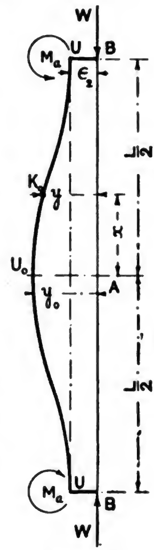



FIG. 16.

| $\frac{W}{P_2}$ | $\sqrt{\frac{W}{P_2}}$ | $\pi\sqrt{\frac{W}{P_2}}$ | $\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}$ | $\cot\pi\sqrt{\frac{W}{P_2}}$ | $\frac{P_2}{\pi 2W}\left(\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}-1\right)$ | $\frac{P_2}{\pi 2W}\left(1-\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right)$ | $\frac{1}{\pi}\frac{P_2}{P_2-4W}\left(\frac{\pi}{2}-\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}\right)$ | $\frac{1}{\pi}\frac{P_2}{P_2-4W}\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0° 0′ | α | α | ·16667 | ·33333 | ·18182 | ·31818 |
| ·1 | ·3162 | 56 55 | 1·1935 | 0·6515 | ·18805 | ·35746 | ·20435 | ·34335 |
| ·2 | ·4472 | 80 29·8 | 1·0139 | 0·1674 | ·21500 | ·38745 | ·23300 | ·37430 |
| ·3 | ·5477 | 98 35·2 | 1·0113 | − 0·1510 | ·25026 | ·42548 | ·26945 | ·41353 |
| ·4 | ·6325 | 113 51 | 1·0934 | − 0·4421 | ·29705 | ·47583 | ·31932 | ·46606 |
| ·5 | ·7071 | 127 16·7 | 1·2568 | − 0·7612 | ·36311 | ·54529 | ·38869 | ·53824 |
| ·6 | ·7746 | 139 25·7 | 1·5376 | − 1·1675 | ·46300 | ·64864 | ·49361 | ·64597 |
| ·7 | ·8367 | 150 36·4 | 2·0374 | − 1·7752 | ·62865 | ·82017 | ·66711 | ·82518 |
| ·8 | ·8944 | 160 59·5 | 3·0702 | − 2·9028 | ·96592 | 1·15966 | 1·02088 | 1·18010 |
| ·9 | ·9487 | 170 46 | 6·2323 | − 6·1515 | 1·97856 | 2·17659 | 2·08175 | 2·24456 |
| 1·0 | 1·0000 | 180 0 | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |

Then
$$\frac{d^2y}{dx^2} + a^2y + \frac{M_a}{W} a^2 = O,$$

to which the solution is

$$y + \frac{M_a}{W} = m \sin ax + n \cos ax.$$

When $x = O$, $\frac{dy}{dx} = O$, and $m = O$;

hence,
$$y + \frac{M_a}{W} = n \cos ax, \text{ and } \frac{dy}{dx} = -an \sin ax.$$

But when $x = \pm\frac{L}{2}$, $\frac{dy}{dx} = O$. Therefore either $n = O$ or $\sin\frac{aL}{2} = O$. If $n = O$, $\frac{dy}{dx} = O$ for all values of $x$, and the column remains straight. If the column bend, $\sin\frac{aL}{2} = O$, and
$$W = \frac{\frac{4\pi^2EI}{L^2}}{1 + \frac{4\pi^2I}{aL^2}} = P_2 \text{ (approximately)} \quad . \quad . \quad . \quad (133)$$
as in Case II, Variation 1.

The load under which an eccentrically loaded position- and direction-fixed column will bend is therefore exactly the same as that for a concentrically loaded column of the same type. The bending moment due to the eccentricity of loading merely increases the value of the fixing moments at the ends.

It is evident that eccentricity of loading will not explain the vagaries of the direction-fixed column.

## Variation 4. The Non-homogeneous Column

The column is of uniform cross section, and originally straight. The load is applied at the centres of area of the end cross sections, and in the direction of the unstrained central axis. The ends are held fixed in their original position and direction. The modulus of elasticity will be assumed to vary.

First suppose that the modulus is constant in any one layer of fibres, but that its value is different in different layers; that is to say, that it varies in the direction of the width but not in the direction of the length.

Such a variation has been shown, in Case I, Variation 4, to be equivalent to a virtual eccentricity of loading, and it follows, as in the preceding Variation, that the crippling load is

$$W = \frac{\frac{4\pi^2EI}{L^2}}{1 + \frac{4\pi^2I}{aL^2}}.$$

(E in this case is $E_a$, the "average" value of the modulus.)

Hence direction-fixing the ends of the column absolutely neutralizes the effect of such a variation in the modulus of elasticity as has been here supposed. The fixing moment will be increased by the amount $W\epsilon_2$.

Secondly, suppose that the modulus varies both in the direction of the length and the width. This will be equivalent to a curvature of the line of resistance, which will, for convenience, be assumed to be a smooth plane curve passing through the centres of area of the end cross sections, and parabolic in shape.

If now, as in Case I, it be further assumed that the modulus at the centre on the concave side of the column to the right of the principal axis $CC_0C$, Fig. 7, is constant and equal to $E_2$, and that the modulus to the left of that line is also constant, but equal to $E_1$, then as before

$$\epsilon_1 = e\,\frac{a_1 \bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

$S = E_a I$ very nearly, and $s_a = \dfrac{W}{E_a a}$. Further, these last two quantities are, practically speaking, constant.

The formulæ of Case II, Variation 2, will apply, and

$$y = \frac{4\epsilon_1}{\alpha L \sin \frac{\alpha L}{2}}\left(\cos \alpha x - \cos \frac{\alpha L}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

$$y_0 = \frac{4\epsilon_1}{\alpha L}\left(\frac{1 - \cos \frac{\alpha L}{2}}{\sin \frac{\alpha L}{2}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

$$M_a = -\frac{8\epsilon_1 W}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot \frac{\alpha L}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

where $\alpha^2 = \dfrac{W}{S(1 - s_a)}$.

The maximum compressive stress at the centre from equation (106) is

$$f_c = -E_2\left(u_2 \frac{W y_0 + M_a}{S} + s_a\right).$$

Neglecting the negative sign of compression,

$$f_c = E_2\left[(v_2 + \epsilon_1)\frac{1}{S}\cdot\frac{8\epsilon_1 W}{\alpha^2 L^2}\left\{\frac{\alpha L}{2}\operatorname{cosec}\frac{\alpha L}{2} - 1\right\} + s_a\right]$$

$$= E_2\left[(v_2 + \epsilon_1)\frac{8\epsilon_1(1 - s_a)}{WL^2}\left\{\frac{\alpha L}{2}\operatorname{cosec}\frac{\alpha L}{2} - 1\right\} + s_a\right] \quad . \quad (138)$$

Or making the same approximations as in previous cases,

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + (v_2 + \epsilon_1)\frac{\epsilon_1}{\kappa^2}\cdot\frac{2P_2}{\pi^2 W}\left\{\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right\}\right]. \quad (139)$$

The maximum compressive stress at the ends, from equation (106), is

$$f_c = -E\left(-u_1\frac{M_a}{S} + s_a\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

Since the line of resistance passes through the centre of area of the end cross section, the value of the modulus must be constant all over that section.

Its magnitude may be taken as equal to $E_a$. Then $u_1 = v_1$ and $S = E_a I$. Hence the maximum compressive stress is

$$f_c = E_a\left\{-v_1\frac{M_a}{E_a I} + \frac{W}{E_a a}\right\}$$

$$= \frac{W}{a}\left[1 + \frac{8\epsilon_1 v_1}{\kappa^2 \alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

$$= \frac{W}{a}\left[1 + \frac{\epsilon_1 v_1}{\kappa^2}\cdot\frac{2P_2}{\pi^2 W}\left(1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right)\right]. \quad . \quad . \quad . \quad (142)$$

Comparing the maximum compressive stress at the centre and ends, it may be shown, as in Variation 2, that in practical columns $\left(\frac{W}{P_2} = 0 \text{ to } \frac{1}{4}\right)$, unless the value of $\frac{v_2 + \epsilon_1}{v_1}$ be greater than 2, the maximum stress will occur at the ends. As the load approaches Euler's limit in value, the stresses in symmetrical sections will become more and more nearly equal, and in unsymmetrical sections will be greater or less at the centre than at the ends, depending on whether $(v_2 + \epsilon_1)$ be greater or less than $v_1$.

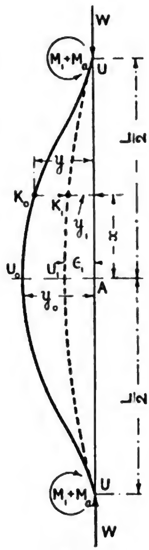


FIG. 17.

### VARIATION 5. COLUMN WITH INITIAL BENDING MOMENTS AT ITS ENDS

The column is of uniform cross section and originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and along the line joining its points of application.

It will be supposed that small initial bending moments $M_1$ are applied to the ends of the column during the process of fixing the ends. These moments will produce an initial curvature which will be circular.

Since E is constant, it follows, as in Variation 1, that $s_a = \frac{W}{Ea}$, $S = EI = \text{const.}$, the central axis will be the line of resistance, and I the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending.

Then if $\epsilon_1$ be the original deflection due to the moments $M_1$,

$$\frac{1}{\rho_1} = \frac{8\epsilon_1}{L^2 + 4\epsilon_1^2} = \frac{M_1}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

$$M_1 = \frac{8\epsilon_1 EI}{L^2 + 4\epsilon_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

Let $UU_1U$ (Fig. 17) be the shape of the line of resistance (the central axis) under the action of the moments $M_1$. Take origin at A in the line of action

of W. $AU = \frac{L}{2}$. Let $x$ and $y_1$ be the co-ordinates of any point $K_1$ in the line of resistance. Then

$$\frac{1}{\rho_1} = -\frac{d^2y_1}{dx^2} = \frac{M_1}{EI} = \frac{8\epsilon_1}{L^2 + 4\epsilon_1^2}$$

and
$$\frac{dy_1}{dx} = -\frac{8\epsilon_1 x}{L^2 + 4\epsilon_1^2}$$
$$= -\frac{4\epsilon_1 L}{L^2 + 4\epsilon_1^2},$$

when $x = \frac{L}{2}$.

Suppose that, after application of the loads WW, the shape of the line of resistance is $UU_0U$, the point $K_1$ becoming the point $K_0$, of which the co-ordinates are $x$ and $y$. The moments at the ends of the column will be increased by $M_a$, the fixing moments, and become $M_1 + M_a$. It is assumed for convenience that the moments at each end of the column are equal. Then the bending moment producing the change of curvature $\frac{d^2}{dx^2}(y - y_1)$ is $(Wy + M_a + M_1) - M_1 = Wy + M_a$, and from equation (105)

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a}{I\left(E - \frac{W}{a}\right)} = 0,$$

or
$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2 + 4\epsilon_1^2} + \frac{Wy + M_a}{I\left(E - \frac{W}{a}\right)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (145)$$

to which the solution is

$$y + \frac{8\epsilon_1}{\alpha^2(L^2 + 4\epsilon_1^2)} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{\alpha^2(L^2 + 4\epsilon_1^2)} + \frac{M_a}{W}\right)\cos \alpha x,$$

where $\alpha^2 = \frac{W}{I\left(E - \frac{W}{a}\right)}$ (compare Variation 2).

Since the ends of the column are fixed both in position and direction, when $x = \frac{L}{2}$, $y = 0$, and $\frac{dy}{dx} = \frac{dy_1}{dx} = -\frac{4\epsilon_1 L}{L^2 + 4\epsilon_1^2}$.
Hence it follows that

$$M_a = -\left[\frac{8\epsilon_1 W}{\alpha^2(L^2 + 4\epsilon_1^2)}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right)\right] \quad . \quad . \quad . \quad . \quad (146)$$

$$y_0 = \frac{4\epsilon_1 L}{\alpha(L^2 + 4\epsilon_1^2)}\left\{\operatorname{cosec}\frac{\alpha L}{2} - \cot\frac{\alpha L}{2}\right\} \quad . \quad . \quad . \quad . \quad (147)$$

and
$$y = \frac{4\epsilon_1 L}{\alpha(L^2 + 4\epsilon_1^2)\sin\frac{\alpha L}{2}}\left\{\cos\alpha x - \cos\frac{\alpha L}{2}\right\} \quad . \quad . \quad . \quad (148)$$

The maximum compressive stress at the centre of the column is obtained from equation (106) and reduces to

$$f_c = \frac{Wv_2}{I}\left(y_0 + \frac{M_a + M_1}{W}\right) + \frac{W}{a}$$

$$= \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_2}{\kappa^2} \cdot \frac{1}{a^2(L^2 + 4\epsilon_1^2)}\left(\frac{aL}{2}\operatorname{cosec}\frac{aL}{2} - 1\right) + \frac{M_1 v_2}{W\kappa^2}\right\}.$$

Now from equation (144)

$$M_1 = \frac{8\epsilon_1 EI}{L^2 + 4\epsilon_1} = \frac{8\epsilon_1 W}{a^2(L^2 + 4\epsilon_1)} \text{ approximately.}$$

Hence
$$f_c = \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_2}{\kappa^2} \cdot \frac{1}{a^2(L^2 + 4\epsilon_1^2)}\left(\frac{aL}{2}\operatorname{cosec}\frac{aL}{2}\right)\right\} \quad . \quad . \quad (149)$$

The maximum compressive stress at the ends of the column is also obtained from equation (106), and reduces to

$$f_c = -\frac{v_1}{I}(M_a + M_1) + \frac{W}{a}$$

$$= \frac{W}{a}\left\{1 + \frac{8\epsilon_1 v_1}{\kappa^2} \cdot \frac{1}{a^2(L^2 + 4\epsilon_1)}\left(1 - \frac{aL}{2}\cot\frac{aL}{2}\right) - \frac{M_1 v_1}{W\kappa^2}\right\}$$

$$= \frac{W}{a}\left\{1 - \frac{8\epsilon_1 v_1}{\kappa^2} \cdot \frac{1}{a^2(L^2 + 4\epsilon_1)}\left(\frac{aL}{2}\cot\frac{aL}{2}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad (150)$$

This equation gives the maximum stress at the ends of the column only when $M_a$ is greater than $M_1$, the condition for which, from equation (146), is

$$\cot\frac{aL}{2} = 0, \text{ or } \frac{aL}{2} = \frac{\pi}{2}, \text{ or } W = \frac{\pi^2 EI}{L^2}.$$

It follows, therefore, that the original bending moment $M_1$ will be greater than the fixing moment $M_a$, until the load reaches Euler's value for the column considered as if it were position-fixed. It is evident, in fact, since $M_1$ and $M_a$ are of necessity opposite in sign, that as $M_a$ grows in value, the total bending moment $(M_a + M_1)$ at the ends of the column will at some moment be zero. It will then be a position-fixed column simply, and since it *ex hypothesi* is no longer straight, the load W holding it in its bent position must have a value at least equal to P, Euler's limit.

Up to this limit the maximum compressive stress occurs of necessity at the centre of the column.

Since $4\epsilon_1^2$ will be small compared with $L^2$, equations (149) and (150) may, if desired, be still further simplified.

## VARIATION 6. THE IMPERFECTLY FIXED COLUMN

The column is of uniform cross section and originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections, and in the original direction of the central axis. The ends are fixed in position and direction, but the direction-fixing is imperfect; that is to say, it will be assumed that a slight angular movement of the ends takes place. Suppose the column to bend.

Since E is constant, it follows that $s_a = \frac{W}{Ea}$, $S = EI =$ const., the central axis will be the line of resistance, and I the moment of inertia of the cross section about the principal axis perpendicular to the plane of bending.

Let $UU_0U$, Fig. 18, be the shape of the line of resistance of the bent column. Take origin at A, $AU = \frac{L}{2}$. Let $x$ and $y$ be the co-ordinates of any point $K_0$ in the line of resistance. From the symmetry of the figure $M_a = M_b$, and hence, from equation (105),

$$\frac{d^2y}{dx^2} + \frac{Wy + M_a}{EI\left(1 - \frac{W}{Ea}\right)} = 0 \quad . \quad . \quad . \quad (151)$$

to which the solution is, as before,

$$y + \frac{M_a}{W} = m \sin ax + n \cos ax.$$

When $x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$. Hence

$$y + \frac{M_a}{W} = n \cos ax \quad . \quad . \quad . \quad . \quad (152)$$

and
$$\frac{dy}{dx} = -an \sin ax.$$

Let it be assumed that, due to a want of rigidity in the end fixings, or otherwise, the ends of the column rotate through a small angle $\sigma$, that is to say, the slope at the ends of the column becomes $-\sigma$. Then

$$\frac{dy}{dx} = -an \sin ax = -\sigma \text{ and } n = \frac{\sigma}{a} \operatorname{cosec} \frac{aL}{2}.$$

Equation (152) becomes, then, $y + \frac{M_a}{W} = \frac{\sigma}{a} \operatorname{cosec} \frac{aL}{2} \cos ax$, but when $x = \frac{L}{2}$, $y = 0$, hence

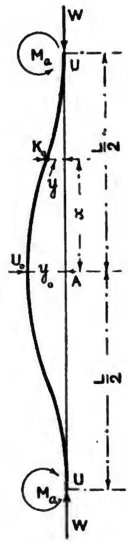


FIG. 18.

$$\frac{M_a}{W} = \frac{\sigma}{a} \cot \frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

$$y = \frac{\sigma}{a}\left\{\frac{\cos ax - \cos \frac{aL}{2}}{\sin \frac{aL}{2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

and
$$y_0 = \frac{\sigma}{a}\left\{\frac{1 - \cos \frac{aL}{2}}{\sin \frac{aL}{2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

The maximum compressive stress at the centre of the column is obtained from equation (106), and reduces to

$$f_c = \frac{Wv_2}{I}\left(y_0 + \frac{M_a}{W}\right) + \frac{W}{a} = \frac{W}{a}\left\{1 + \frac{v_2}{\kappa^2}\cdot\frac{\sigma}{\alpha}\operatorname{cosec}\frac{\alpha L}{2}\right\} \quad . \quad . \quad (156)$$

At the ends of the column the expression becomes

$$f_c = -\frac{v_1}{I}M_a + \frac{W}{a} = \frac{W}{a}\left\{1 - \frac{v_1}{\kappa^2}\cdot\frac{\sigma}{\alpha}\cot\frac{\alpha L}{2}\right\}. \quad . \quad . \quad (157)$$

Comparing these equations with those obtained for Case II, Variation 2, it is evident that equations (119) and (154), (117) and (155), are alike in form; in fact, if the value $\sigma = \frac{4\epsilon_1}{L}$ from equation (114) be introduced into equations (154) and (155), they become identical with equations (119) and (117). It is evident that, so far as the deflections produced are concerned, the effect of want of rigidity of the end fixings is equivalent to that produced by an initial deflection

$$\epsilon_1 = \frac{\sigma L}{4}$$

Comparing, however, the maximum compressive stresses at the centre, equations (120) and (156), it is evident that this stress in Variation 6 is greater than that in Variation 2 for an initial deflection $\epsilon_1 = \frac{\sigma L}{4}$. This might have been conjectured, for since the direction-fixing is imperfect, the fixing moment at the ends will be reduced, and hence the bending moment at the centre increased.

Turning next to the maximum compressive stress at the ends, equation (157) calls for some comment. It appears from the negative sign that the effect of the bending moment is to decrease the compressive stress on the layer of fibres $v = v_1$ up to the point where $\frac{\alpha L}{2} = \frac{\pi}{2}$, and then to increase it. This seems somewhat surprising. Some light is thrown on the problem by equation (153). If $M_a$ be put equal to zero, $\cot\frac{\alpha L}{2} = 0$, and $\frac{\alpha L}{2} = \frac{\pi}{2}$. But if $\alpha L = \pi$, $W = \frac{\pi^2 EI}{L^2}$, that is to say, Euler's crippling load for the column, supposing it to be merely position-fixed at the ends. Hence it appears that the column will remain perfectly straight until $W = \frac{\pi^2 EI}{L^2}$, when it will commence to bend. At first, and until the slope at the ends $= \sigma$, it will behave as a *position-fixed* column. When the slope $= \sigma$, it will also become direction-fixed, and the formulæ obtained above will be applicable. That is to say, these formulæ are only applicable so long as $W > \frac{\pi^2 EI}{L^2}$. Hence the negative sign in equation (157) does not, in reality, represent a reduction in the compressive stress. Up to this point $f_c = f_a = \frac{W}{a}$.

If now in equation (122) $\epsilon_1$ be put $= \frac{\sigma L}{4}$, it is evident from a comparison

of equation (122) with equation (157) that the stress $f_c$ in Variation 6 is less than that in Variation 2.

Had the more general case of a column with initial curvature and imperfect direction-fixing been considered, equations (153), (154), and (155) would become

$$\frac{M_a}{W} = -\frac{8\epsilon_1}{a^2L^2} + \left(\frac{4\epsilon_1}{aL} + \frac{\sigma}{a}\right) \cot \frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (158)$$

$$y_0 = \left(\frac{4\epsilon_1}{aL} + \frac{\sigma}{a}\right) \left\{ \frac{1 - \cos \dfrac{aL}{2}}{\sin \dfrac{aL}{2}} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

$$y = \left(\frac{4\epsilon_1}{aL} + \frac{\sigma}{a}\right) \left\{ \frac{\cos ax - \cos \dfrac{aL}{2}}{\sin \dfrac{aL}{2}} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

At the centre

$$f_c = \frac{W}{a}\left[1 + \frac{v_2}{\kappa^2}\left\{\left(\frac{4\epsilon_1}{aL} + \frac{\sigma}{a}\right) \operatorname{cosec} \frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2}\right\}\right] \quad . \quad . \quad (161)$$

At the ends

$$f_e = \frac{W}{a}\left[1 - \frac{v_1}{\kappa^2}\left\{\left(\frac{4\epsilon_1}{aL} + \frac{\sigma}{a}\right) \cot \frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2}\right\}\right]. \quad . \quad . \quad (162)$$

These equations which, as they stand, are applicable to all values of W, evidently reduce to those given above if $\epsilon_1 = 0$, or to those of Variation 2 if $\sigma = 0$.

It may be well to note that the whole of the above equations hold if $\sigma$ be a function increasing with W.

## Variation 7. The Ordinary Column

In addition to the various imperfections in the conditions which tend to produce flexure in the ordinary column with position-fixed ends (see p. 40), there is, in the case of the column with position- and direction-fixed ends, the effect of imperfect direction-fixing to be taken into account.

As in Case I, Variation 6, the whole of these imperfections except the last can be accounted for by assuming the column to be both initially curved and eccentrically loaded. The imperfection in the direction-fixing may be allowed for by assuming a slight increase in the slope at the ends.

For the reasons given in Case I, it will be assumed that all the imperfections tend to produce flexure in the plane perpendicular to the principal axis of elasticity about which S is a minimum. The bending will then be uniplanar.

The column is assumed to be of uniform cross section. Let $VV_1V$, Fig. 19, be the original shape of the central axis, BB the line of action of the load, and $UU_1U$ the original shape of the line of resistance. Then UB is the eccentricity of the load $= \epsilon_2$. Of this eccentricity $VB = \epsilon_4$ is due to inaccurate centering, and $UV = \epsilon_6$ to variations in the modulus of elasticity, $\epsilon_2 = \epsilon_4 + \epsilon_6$.

$U_1U'$ is the original deflection of the line of resistance $= \epsilon_1$. Of this deflection $U'U'' = V_1V' = \epsilon_3$ is due to the original deflection of the central axis, and $U_1U'' = \epsilon_5$ is the original deflection due to variations in the modulus of elasticity, $\epsilon_1 = \epsilon_3 + \epsilon_5$.

It will be assumed that the curve $UU_1U$ is a smooth plane curve, and an arc of a parabola, its exact shape not being of great importance. Let $UU_0U$ be the final shape of the line of resistance. Take origin at A. $AB = \frac{L}{2}$. Let $x$ and $y$ be the co-ordinates of any point $K_0$ on the line of resistance in its final position, and suppose that $K_1$ was the original position of this point, and $x$ and $y_1$ its co-ordinates.

Then the equation to the line $UU_1U$ is

$$y_1 = \epsilon_2 + \epsilon_1 \left(1 - \frac{4x^2}{L^2}\right) \quad . \quad (163)$$

Hence $\frac{dy_1}{dx} = -\frac{8\epsilon_1 x}{L^2}$ and $\frac{d^2y_1}{dx^2} = -\frac{8\epsilon_1}{L^2}$.

When $x = \frac{L}{2}$, $\frac{dy_1}{dx} = -\frac{4\epsilon_1}{L}$.

Since by the symmetry of the figure $M_a = M_b$, equation (105) becomes

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a}{S(1 - s_a)} = 0$$

or

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + \frac{Wy + M_a}{S(1 - s_a)} = 0 \quad . \quad (164)$$

As before S and $s_a$ will be assumed to be constant. Let $a^2 = \frac{W}{S(1 - s_a)}$. Then the solution to the differential equation is

$$y = m \sin ax + n \cos ax - \frac{8\epsilon_1}{a^2L^2} - \frac{M_a}{W}.$$

When $x = 0$, $\frac{dy}{dx} = 0$ and $m = 0$; also $y = y_0$.

Hence $$n = y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}$$

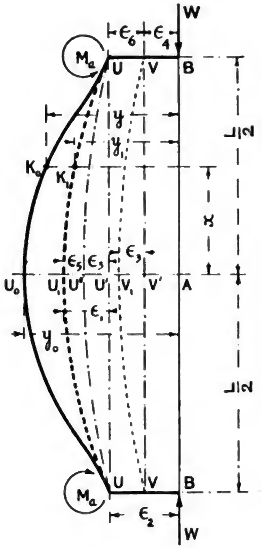



FIG. 19.

and $$y + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}\right) \cos ax \quad . \quad . \quad . \quad . \quad (165)$$

Now the ends of the column are fixed both in position and direction, but the direction-fixing is imperfect. The original slope at the ends, where $x = \frac{L}{2}$, was $-\frac{4\epsilon_1}{L}$. Suppose this increases to $-\frac{4k\epsilon_1}{L}$ where $k$ is a coefficient somewhat greater than unity. The value of $k$ will probably vary with the magnitude of the load, but for the purposes of this analysis $k$ is a constant.

When $x = \frac{L}{2}$, $y = \epsilon_2$ and $\frac{dy}{dx} = -\frac{4k\epsilon_1}{L}$. Hence, from equation (165),

$$\epsilon_2 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}\right) \cos \frac{aL}{2}$$

and
$$+\frac{4k\epsilon_1}{L} = a\left(y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}\right) \sin \frac{aL}{2}$$

or
$$\epsilon_2 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = \frac{4k\epsilon_1}{aL} \cot \frac{aL}{2}$$

whence
$$M_a = -\frac{8W\epsilon_1}{a^2L^2}\left\{1 - k \cdot \frac{aL}{2} \cot \frac{aL}{2}\right\} - W\epsilon_2 \quad . \quad . \quad . \quad . \quad (166)$$

$$y_0 = \frac{4k\epsilon_1}{aL}\left(\frac{1 - \cos \frac{aL}{2}}{\sin \frac{aL}{2}}\right) + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167)$$

$$= \frac{4k\epsilon_1}{aL} \cdot \tan \frac{aL}{4} + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168)$$

$$y = \frac{4k\epsilon_1}{aL}\left(\frac{\cos ax - \cos \frac{aL}{2}}{\sin \frac{aL}{2}}\right) + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

and
$$\Delta = y_0 - \epsilon_1 - \epsilon_2 = \frac{4k\epsilon_1}{aL}\left(\frac{1 - \cos \frac{aL}{2}}{\sin \frac{aL}{2}}\right) - \epsilon_1 \quad . \quad . \quad . \quad . \quad (170)$$

It will be observed that the effect of the imperfect direction-fixing is to increase the value of $\epsilon_1$ to $k\epsilon_1$, compare equations (119) and (169). The value of $\Delta$, the deflection actually produced by the load, is solely dependent on $k$ and $\epsilon_1$.

Approximate values for $M_a$ and $y_0$ may be obtained as follows:

$$a^2 = \frac{W}{S(1 - s_a)} = \frac{W}{E_aI\left(1 - \frac{W}{E_aa}\right)} = \frac{W}{E_aI}.$$

Hence
$$aL = 2\pi\sqrt{\frac{W}{P_2}} \text{ and } \frac{aL}{2} = \pi\sqrt{\frac{W}{P_2}}$$

approximately, and

$$M_a = -\frac{8\epsilon_1E_aI}{L^2}\left(1 - k\pi\sqrt{\frac{W}{P_2}} \cot \pi\sqrt{\frac{W}{P_2}}\right) - W\epsilon_2 \quad . \quad . \quad . \quad . \quad (171)$$

$$y_0 = \epsilon_2 + \frac{2k\epsilon_1}{\pi\sqrt{\frac{W}{P_2}}}\left\{\frac{1 - \cos \pi\sqrt{\frac{W}{P_2}}}{\sin \pi\sqrt{\frac{W}{P_2}}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

The maximum compressive stress at the centre on the concave side of the column is, from equation (18),

$$f_c = -\,E_2\left(u_2\,\frac{M}{S} + s_a\right)$$

Neglecting the negative sign of compression and putting $u_2 = v_2 + \epsilon_5 + \epsilon_6$,

$$f_c = E_2\left[(v_2 + \epsilon_5 + \epsilon_6)\,\frac{Wy_0 + M_a}{S} + s_a\right]$$

which by equations (166) and (167) becomes

$$f_c = E_2\left[(v_2 + \epsilon_5 + \epsilon_6)\,\frac{W}{S}\cdot\frac{8\epsilon_1}{a^2L^2}\left\{k\cdot\frac{aL}{2}\operatorname{cosec}\frac{aL}{2} - 1\right\} + s_a\right] \quad . \quad . \quad (173)$$

If, now, as in Case I, it be assumed that in any cross section the modulus of elasticity on the concave side of the column to the right of the principal axis $CC_0C$, Fig. 7, is constant, and that the modulus on the convex side to the left of that axis is also constant, but different in value, then the axis $CC_0C$ will be perpendicular to the plane of flexure, and it may be shown as before that if $E_1$ and $E_2$ be the greatest and least values of the modulus at the central cross section, then [equation (88)]

$$\epsilon_5 + \epsilon_6 = e\,\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

where $e = \dfrac{E_1 - E_2}{E_a}$, the fractional variation of the modulus of elasticity.

Similarly $\epsilon_5 = e_5\dfrac{a_1\bar{v}_1}{a}$, and $\epsilon_6 = e_6\dfrac{a_1\bar{v}_1}{a}$ where $e_5$ and $e_6$ are the fractional variations of the modulus corresponding to $\epsilon_5$ and $\epsilon_6$. Then $e = e_5 + e_6$ and

$$\epsilon_1 = \epsilon_3 + \epsilon_5 = \epsilon_3 + e_5\,\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

$$\epsilon_2 = \epsilon_4 + \epsilon_6 = \epsilon_4 + e_6\,\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Making the same approximations as in previous cases,

$$f_c = E_a\left(1 - \frac{e}{2}\right)\left[\left(v_2 + e\,\frac{a_1\bar{v}_1}{a}\right)\frac{W}{E_aI}\cdot\frac{2\epsilon_1P_2}{\pi^2W}\left\{k\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right\} + \frac{W}{E_aa}\right] \quad (174)$$

$$= \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\,\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2}\cdot\frac{P_2}{W}\left\{k\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right\}\right] (175)$$

The maximum compressive stress at the ends of the column is obtained from equation (16):

$$f = E\left\{\frac{Mu}{S} - s_a\right\}.$$

Now on the concave side of the column at the ends $u = u_1 = v_1 - \epsilon_6$; suppose the value of E to be $E_1'$. The value of the bending moment at the ends is $W\epsilon_2 + M_a$. Hence,

$$f_c = -\,E_1'\left[-\,\frac{W\epsilon_2 + M_a}{S}(v_1 - \epsilon_6) + s_a\right].$$

Neglecting the negative sign of compression, and inserting the value of $M_a$ from equation (166),

$$f_c = E_1' \left[ (v_1 - \epsilon_6) \frac{8W\epsilon_1}{Sa^2L^2} \left\{ 1 - k \frac{aL}{2} \cot \frac{aL}{2} \right\} + s_a \right]$$

But $E_1' = E_a\left(1 + \frac{e_6}{2}\right)$ approximately, and $\epsilon_6 = e_6 \frac{a_1\bar{v}_1}{a}$. Hence,

$$f_c = E_a\left(1 + \frac{e_6}{2}\right)\left[\left(v_1 - e_6 \frac{a_1\bar{v}_1}{a}\right)\frac{W}{E_aI} \cdot \frac{2\epsilon_1 P_2}{\pi^2 W} \right.$$

$$\left. \times \left\{1 - k\pi\sqrt{\frac{W}{P_2}} \cot \pi\sqrt{\frac{W}{P_2}}\right\} + \frac{W}{E_a a}\right] \quad . \quad . \quad (176)$$

$$= \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[1 + \left(v_1 - e_6 \frac{a_1 v_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2} \cdot \frac{P_2}{W} \right.$$

$$\left. \times \left\{1 - k\pi\sqrt{\frac{W}{P_2}} \cot \pi\sqrt{\frac{W}{P_2}}\right\}\right] \quad . \quad . \quad . \quad . \quad (177)$$

These equations may be simplified for practical use in a manner similar to that adopted in Case I, Variation 6. From Fig. 15 it will be observed that for all practical cases, that is to say, for those in which the ratio $\frac{W}{P_2}$ varies from 0 to $\frac{1}{4}$, the curves are very nearly straight lines. The functions

$$\frac{1}{\pi^2}\frac{P_2}{W}\left\{\pi\sqrt{\frac{W}{P_2}} \operatorname{cosec} \pi\sqrt{\frac{W}{P_2}} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{curve 1}$$

and

$$\frac{1}{\pi^2}\frac{P_2}{W}\left\{1 - \pi\sqrt{\frac{W}{P_2}} \cot \pi\sqrt{\frac{W}{P_2}}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{curve 2}$$

may with safety be replaced by the straight lines

$$0{\cdot}17 + 0{\cdot}26\frac{W}{P_2}\text{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

and

$$0{\cdot}33 + 0{\cdot}29\frac{W}{P_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179)$$

respectively. Now equation (175) may be written

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2}\frac{P_2}{W}k \right.$$

$$\left. \times \left\{\pi\sqrt{\frac{W}{P_2}} \operatorname{cosec} \pi\sqrt{\frac{W}{P_2}} - 1 + 1 - \frac{1}{k}\right\}\right]$$

$$= \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{2k\epsilon_1}{\kappa^2} \right.$$

$$\left. \times \left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{1}{\pi^2} \cdot \frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right] \quad . \quad . \quad (180)$$

* If preferred, the straight line $0{\cdot}18 + 0{\cdot}28\frac{W}{P_2}$, corresponding to the sinusoidal curve 3, might be used.

Neglecting the two factors containing $e$, since, as was shown in Case I, Variation 6, the error introduced thereby is small, the above equation reduces to

$$f_c = \frac{W}{a}\left[1 + \frac{2k\epsilon_1 v_2}{\kappa^2}\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{1}{\pi^2}\cdot\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right] \quad . \quad (181)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 v_2}{\kappa^2}\left\{1{\cdot}44\,k - 1\right\}\right] \quad . \quad . \quad . \quad . \quad . \quad (182)$$

If the direction-fixing be perfect, $k = 1$, and

$$f_c = \frac{W}{a}\left[1 + 0{\cdot}44\,\frac{\epsilon_1 v_2}{\kappa^2}\right] = \frac{W}{a}\left[1 + 0{\cdot}44\,\frac{\epsilon_1}{\omega_2}\right] \quad . \quad . \quad . \quad (183)$$

Equation (177) for the maximum compressive stress at the ends of the column may likewise be simplified. As before, the two terms containing $e_6$ may be neglected, and the equation written

$$f_c = \frac{W}{a}\left[1 + v_1\frac{2k\epsilon_1}{\pi^2\kappa^2}\cdot\frac{P_2}{W}\left\{\frac{1}{k} - 1 + 1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right\}\right]$$

$$= \frac{W}{a}\left[1 + \frac{2k\epsilon_1 v_1}{\kappa^2}\left\{0{\cdot}33 + 0{\cdot}29\frac{W}{P_2} - \frac{1}{\pi^2}\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right] \quad . \quad . \quad . \quad (184)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 v_1}{\kappa^2}\left\{1 - 0{\cdot}22\,k\right\}\right] \quad . \quad . \quad . \quad . \quad . \quad (185)$$

The worst possible assumption which can be made regarding the value of $k$ in equations (184) and (185) is that it is equal to unity, that is to say, that the column is perfectly direction-fixed, for it is evident that any yielding of the end connexions decreases the stress there. It would be more logical, perhaps, to use the same value for $k$ at the ends as at the middle; it is safer to put $k = 1$ at the ends. In this case equation (184) becomes

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 v_1}{\kappa^2}\left\{0{\cdot}66 + 0{\cdot}58\frac{W}{P_2}\right\}\right] \quad . \quad . \quad . \quad . \quad (186)$$

and equation (185), $\left(\frac{W}{P_2} = \frac{1}{5}\right)$,

$$f_c = \frac{W}{a}\left[1 + 0{\cdot}78\,\frac{\epsilon_1 v_1}{\kappa^2}\right] = \frac{W}{a}\left[1 + 0{\cdot}78\,\frac{\epsilon_1}{\omega_1}\right] \quad . \quad . \quad . \quad (187)$$

The maximum tensile stress on the convex side at the centre of the column is, from equation (17),

$$f_t = E_1\left\{u_1\frac{M}{S} - s_a\right\}.$$

Now $u_1 = v_1 - \epsilon_5 - \epsilon_6$, and $M = Wy_o + M_a$,

$$f_t = E_1\left[(v_1 - \epsilon_5 - \epsilon_6)\frac{Wy_o + M_a}{S} - s_a\right].$$

Introducing the value of the bending moment from equations (166) and (167),

$$f_t = \mathrm{E}_1\left[(v_1 - \epsilon_5 - \epsilon_6)\frac{\mathrm{W}}{\mathrm{S}}\frac{8\epsilon_1}{a^2\mathrm{L}^2}\left\{k\frac{a\mathrm{L}}{2}\operatorname{cosec}\frac{a\mathrm{L}}{2} - 1\right\} - s_a\right] \quad . \quad (188)$$

which becomes as before

$$f_t = \mathrm{E}_a\left(1+\frac{e}{2}\right)\left[\left(v_1 - e\frac{a_1\bar{v}_1}{a}\right)\frac{\mathrm{W}}{\mathrm{E}_a\mathrm{I}}\frac{2\epsilon_1\mathrm{P}_2}{\pi^2\mathrm{W}}\right.$$
$$\left.\times\left\{k\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\operatorname{cosec}\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}} - 1\right\} - \frac{\mathrm{W}}{\mathrm{E}_a a}\right] \quad . \quad (189)$$

$$= \frac{\mathrm{W}}{a}\left(1+\frac{e}{2}\right)\left[\left(v_1 - e\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2}\cdot\frac{\mathrm{P}_2}{\mathrm{W}}\right.$$
$$\left.\times\left\{k\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\operatorname{cosec}\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}} - 1\right\} - 1\right] \quad . \quad . \quad (190)$$

The maximum tensile stress on the convex side at the ends of the column is, from equation (16),

$$f = \mathrm{E}\left\{\frac{\mathrm{M}u}{\mathrm{S}} - s_a\right\}$$

Now on the convex side of the column at the ends $u = -u_2 = -(v_2 + \epsilon_6)$ and $\mathrm{E} = \mathrm{E}_2' = \mathrm{E}_a\left(1 - \frac{e_6}{2}\right)$. Hence

$$f_t = \mathrm{E}_2'\left[(v_2+\epsilon_6)\frac{\mathrm{W}\epsilon_2 + \mathrm{M}_a}{\mathrm{S}} - s_a\right]$$

$$= \mathrm{E}_2'\left[(v_2+\epsilon_6)\frac{8\mathrm{W}\epsilon_1}{\mathrm{S}a^2\mathrm{L}^2}\left\{1 - k\frac{a\mathrm{L}}{2}\cot\frac{a\mathrm{L}}{2}\right\} - s_a\right] \quad . \quad (191)$$

Making the same approximations as before,

$$f_t = \mathrm{E}_a\left(1-\frac{e_6}{2}\right)\left[\left(v_2 - e_6\frac{a_1\bar{v}_1}{a}\right)\frac{\mathrm{W}}{\mathrm{E}_a\mathrm{I}}\cdot\frac{2\epsilon_1\mathrm{P}_2}{\pi^2\mathrm{W}}\right.$$
$$\left.\times\left\{1 - k\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\cot\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\right\} - \frac{\mathrm{W}}{\mathrm{E}_a a}\right] \quad . \quad . \quad (192)$$

$$f_t = \frac{\mathrm{W}}{a}\left(1-\frac{e_6}{2}\right)\left[\left(v_2 + e_6\frac{a_1 v_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2}\cdot\frac{\mathrm{P}_2}{\mathrm{W}}\right.$$
$$\left.\times\left\{1 - k\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\cot\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\right\} - 1\right] . \quad . \quad .. (193)$$

Equations (190) and (193) may be simplified for practical use in the same manner as before. The terms containing $e$ may be neglected, though, as stated in Case I, Variation 6, this is not so satisfactory an approximation as in the equation for $f_c$. Equation (190) may then be written

$$f_t = \frac{\mathrm{W}}{a}\left[\frac{2k\epsilon_1 v_1}{\pi^2\kappa^2}\frac{\mathrm{P}_2}{\mathrm{W}}\left\{\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}}\operatorname{cosec}\pi\sqrt{\frac{\mathrm{W}}{\mathrm{P}_2}} - 1 + 1 - \frac{1}{k}\right\} - 1\right]$$

$$= \frac{\mathrm{W}}{a}\left[\frac{2k\epsilon_1 v_1}{\kappa^2}\left\{0{\cdot}17 + 0{\cdot}26\frac{\mathrm{W}}{\mathrm{P}_2} + \frac{1}{\pi^2}\cdot\frac{\mathrm{P}_2}{\mathrm{W}}\left(1 - \frac{1}{k}\right)\right\} - 1\right] \quad . \quad . \quad (194)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_t = \frac{W}{a}\left[\left\{1{\cdot}44\,k - 1\right\}\frac{\epsilon_1 v_1}{\kappa^2} - 1\right] \quad . \quad . \quad . \quad . \quad . \quad (195)$$

If the direction-fixing be perfect, $k = 1$, and

$$f_t = \frac{W}{a}\left[0{\cdot}44\,\frac{\epsilon_1 v_1}{\kappa^2} - 1\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

an expression for the tensile stress at the centre of the column.

Similarly, from equation (193),

$$f_t = \frac{W}{a}\left[\frac{2k\epsilon_1 v_2}{\kappa^2}\left\{0{\cdot}33 + 0{\cdot}29\,\frac{W}{P_2} - \frac{1}{\pi^2}\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\} - 1\right] \quad . \quad (197)$$

from which the tensile stress at the ends of the column may be found. Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_t = \frac{W}{a}\left[\frac{\epsilon_1 v_2}{\kappa^2}\left\{1 - 0{\cdot}22\,k\right\} - 1\right] \quad . \quad . \quad . \quad . \quad . \quad (198)$$

If the direction-fixing be perfect, $k = 1$, and

$$f_t = \frac{W}{a}\left[0{\cdot}78\,\frac{\epsilon_1 v_2}{\kappa^2} - 1\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

A common method of treating the direction-fixed column is to consider it as a position-fixed column of length $q$L, where $q$ is a fraction. The value of $q$ may be determined by finding the points in the length of the column where the bending moment is zero. The expression for the bending moment anywhere, from equations (166) and (169), is

$$Wy + M_a = \frac{4Wk\epsilon_1}{\alpha L}\left(\frac{\cos \alpha x - \cos\frac{\alpha L}{2}}{\sin\frac{\alpha L}{2}}\right) - \frac{8W\epsilon_1}{\alpha^2 L^2}\left(1 - k\,\frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right) \quad . \quad (200)$$

Equating this to zero,

$$\cos \alpha x = \frac{2}{k\alpha L}\sin\frac{\alpha L}{2}$$

where $2x = q$L. Therefore,

$$\cos q\cdot\frac{\alpha L}{2} = \frac{2}{k\alpha L}\sin\frac{\alpha L}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

or

$$\cos q\pi\sqrt{\frac{W}{P_2}} = \frac{1}{k\pi}\sqrt{\frac{P_2}{W}}\sin \pi\sqrt{\frac{W}{P_2}} \quad . \quad . \quad . \quad (202)$$

The value of $q$ is therefore a function of W, and varies with the shape of the cross section and the efficiency of the end connexions. It is independent of $\epsilon_1$ and $\epsilon_2$ except in so far as the value of W is dependent on the former. If the direction-fixing be perfect ($k = 1$), the value of $q$ falls slowly from 0·578 * to 0·561 as $\frac{W}{P_2}$ varies from O to $\frac{1}{4}$. Its value when $W = P_2$ is $\frac{1}{2}$.

* For a direction-fixed *beam* $q = 0{\cdot}578$.

A rough idea as to the magnitude of $k$ may be obtained from equation (202):

$$k = \frac{\frac{1}{\pi}\sqrt{\frac{P_2}{W}} \sin \pi \sqrt{\frac{W}{P_2}}}{\cos q\pi \sqrt{\frac{W}{P_2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (203)$$

If the direction-fixing be perfect, $q$ varies from 0·58 to 0·56 as $\frac{W}{P_2}$ varies from O to $\frac{1}{4}$. If the direction-fixing be so imperfect that the column has entirely lost the benefit of the direction-fixing, and has become a concentrically loaded position-fixed column, $q = 1$. It may be assumed that the fixing moments at the end of the column are sufficient to neutralize the moments $W\epsilon_2$, for if not, the column should be considered merely as a position-fixed eccentrically loaded column, and treated as in Case I, Variation 6.

It is evident, then, that in an imperfectly direction-fixed column, $q$ will lie somewhere between 0·58 and unity when $\frac{W}{P_2} = O$, and between 0·56 and unity when $\frac{W}{P_2} = \frac{1}{4}$. A very safe assumption to make is that it lies midway between these values, that is to say that $q$ for an imperfectly direction-fixed column varies from 0·79 to 0·78 as $\frac{W}{P_2}$ varies from O to $\frac{1}{4}$. In this case it follows from equation (203) that $k$ varies from 1·0 to 1·88. The variations in $q$ and $k$ on the above assumptions are exhibited in the accompanying table and Fig. 15. The values for imperfect direction-fixing must, of course, be regarded merely as indications of the probable magnitude $k$ and $q$. An accurate determination of their values is not possible theoretically, they should be measured in practical cases.

| $\frac{W}{P_2}$ | Perfect Direction-fixing. | | Imperfect Direction-fixing. | |
|---|---|---|---|---|
| | $k$ | $q$ | $k$ | $q$ |
| 0 | 1 | ·578 | 1·0 | ·789 |
| ·1 | 1 | ·571 | 1·19 | ·785 |
| ·2 | 1 | ·564 | 1·54 | ·782 |
| ·25 | 1 | ·561 | 1·88 | ·780 |

If, however, the value of $q$ is as great as 0·78, and, in consequence, the value of $k$ at the limit of working conditions is as large as 1·88, it would appear wiser to limit the working conditions in the case of imperfectly fixed columns to the range

$$\frac{W}{P_1} < \frac{1}{5}$$

where $P_1 = \frac{\pi^2 EI}{(qL)^2}$, instead of the range $\frac{W}{P_2} < \frac{1}{5}$. In this case the approximate

equations (182), (185), (195), and (198) should not be used, but recourse should be made to the more exact expressions from which they are derived.

Another estimate for the value of $f_c$ at the centre of the column, without introducing the value of $k$, may be obtained by substituting the value of $k$ from equation (203) in equation (175). This equation then becomes,

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\,\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1}{\pi^2\kappa^2}\cdot\frac{P_2}{W}\left\{\sec q\pi\sqrt{\frac{W}{P_2}} - 1\right\}\right]$$

or, neglecting as before the two factors containing $e$,

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 v_2}{\kappa^2}\cdot\frac{2P_2}{\pi^2 W}\left\{\sec q\pi\sqrt{\frac{W}{P_2}} - 1\right\}\right]$$

If $\dfrac{\pi^2 EI}{(qL)^2} = P_1$,

$$f_c = \frac{W}{a}\left[1 + q^2\frac{\epsilon_1 v_2}{\kappa^2}\cdot\frac{8P_1}{\pi^2 W}\left\{\sec\frac{\pi}{2}\sqrt{\frac{W}{P_1}} - 1\right\}\right] \quad . \quad . \quad . \quad (204)$$

an equation for the stress at the centre of the column involving $q$ only [compare equation (33)]. This equation has some advantages in that $q$ is very nearly constant under working conditions, whereas $k$ varies considerably.

If under working conditions $\dfrac{W}{P_1} < \dfrac{1}{5}$, it follows from Fig. 5, curve 31, that $\dfrac{8P_1}{\pi^2 W}\left\{\sec\dfrac{\pi}{2}\sqrt{\dfrac{W}{P_1}} - 1\right\}$ may be safely replaced by $\left(1 + 1{\cdot}25\,\dfrac{W}{P_1}\right)$, and therefore equation (204) may be written

$$f_c = \frac{W}{a}\left[1 + q^2\frac{\epsilon_1 v_2}{\kappa^2}\left(1 + 1{\cdot}25\,\frac{W}{P_1}\right)\right] \quad . \quad . \quad . \quad . \quad . (205)$$

a simple equation for the stress at the centre of an imperfectly direction-fixed column, provided that $\dfrac{W}{P_1} < \dfrac{1}{5}$. If in this equation $\dfrac{W}{P_1}$ be given its superior limit $\dfrac{1}{5}$, and $q = 0{\cdot}78$,

$$f_c = \frac{W}{a}\left[1 + 0{\cdot}76\,\frac{\epsilon_1 v_2}{\kappa^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205\text{A})$$

## CASE III. Columns with Flat Ends. Uniplanar Bending

A column with flat ends will act as a direction-fixed column while the contact at the ends is of such a nature that a bending moment may be transmitted from the ends of the column to the surface on which it abuts. Unless and until this condition is fulfilled, the column will act as a position-fixed column. Flat-ended columns have the further peculiarity that no tensile stress can exist on the end cross sections. Should the compressive stress on one edge of the end cross section become zero, any increase in the load will tend to cause the end cross sections of the column to rotate about the opposite edge as a hinge.

It is conceivable, therefore, that during the experimental history of a flat-ended column it may pass from the condition of an eccentrically loaded

position-fixed column to that of a direction-fixed column, returning later to that of an eccentrically loaded position-fixed column again.

### VARIATION 1. IDEAL CONDITIONS

The column is of uniform cross section and originally perfectly straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and in the direction of the unstrained central axis. The end cross sections are perfectly flat and normal to the central axis, and bear on perfectly flat and normal surfaces.

Then it may be shown, exactly as in Case II, Variation 1, that if the column is to bend, the value of the load must be greater than

$$W = \frac{\dfrac{4\pi^2 EI}{L^2}}{1 + \dfrac{4\pi^2 I}{aL^2}}$$

The column, in fact, may be considered as a position- and direction-fixed specimen.

### VARIATION 2. COLUMN WITH INITIAL CURVATURE

The column is of uniform cross section, but not originally straight. The modulus of elasticity is constant and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections and along the line joining its points of application. The end cross sections are perfectly flat, and (*a*) normal to the line of action of the load, and bear on perfectly flat surfaces also normal to that line (Fig. 20); (*b*) normal to the central axis, and bear on perfectly flat surfaces also normal to that axis (Fig. 21).

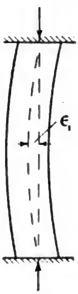


FIG. 20.

FIG. 21.

While the end surfaces remain in contact, the column is evidently under exactly the same conditions as that in Case II, Variation 2, and all the formulæ there obtained will apply. The point at which these cease to hold is obtained by equating the expression for the stress on the convex side of the column at the ends to zero. This stress is obtained from equations (17) and (116), and is

$$f_t = E\left\{\frac{M_a u}{EI} - \frac{W}{Ea}\right\}$$

where $u = -v_2$ and $M_a = -\dfrac{8\epsilon_1 W}{\alpha^2 L^2}\left(1 - \dfrac{\alpha L}{2}\cot\dfrac{\alpha L}{2}\right)$.

Hence $$f_t = \frac{W}{a}\left\{\frac{8\epsilon_1 v_2}{\kappa^2}\cdot\frac{1}{\alpha^2 L^2}\left(1 - \frac{\alpha L}{2}\cot\frac{\alpha L}{2}\right) - 1\right\}.$$

If $f_t =$ zero, $$\cot\frac{\alpha L}{2} = \frac{2}{\alpha L} - \frac{\alpha L\kappa^2}{4\epsilon_1 v_2}$$

or
$$\cot \pi\sqrt{\frac{W}{P_2}} = \frac{1}{\pi}\sqrt{\frac{P_2}{W}} - \frac{2}{2\epsilon_1 v_2}\sqrt{\frac{W}{P_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

an equation from which, for a given specimen, the value of W at which "swinging round" occurs may be obtained. It may be written

$$\cot \frac{L}{2\kappa}\sqrt{\frac{W}{Ea}} = \frac{2\kappa}{L}\sqrt{\frac{Ea}{W}} - \frac{\kappa L}{4\epsilon_1 v_2}\sqrt{\frac{W}{Ea}} \quad . \quad . \quad . \quad . \quad . \quad (207)$$

from which it is evident that the relation between $\frac{L}{\kappa}$ and W when "swinging round" occurs is not a simple one.

### VARIATION 3. THE ECCENTRICALLY LOADED COLUMN

As in Case II, Variation 3, it may be shown that the effect of eccentricity of loading is merely to increase the value of the fixing moments at the ends. The case reduces in fact to Variation 1.

### VARIATION 4. COLUMN WITH IMPERFECT BEARINGS

*Assumption* (*a*).—The column is of uniform cross section, and originally perfectly straight. The modulus of elasticity is constant everywhere and the column is perfectly homogeneous. The column is compressed between two perfectly flat surfaces which always remain parallel to their original position, but which make a small angle $\sigma$ with the end cross sections of the column, which are also perfectly flat surfaces (see Fig. 22). The angular clearance $\sigma$ may be due to inclination of the end surfaces of the column, or to inclination of the abutting surfaces, or to the two causes combined. For convenience it will be assumed that the angle has the same value at each end of the column.

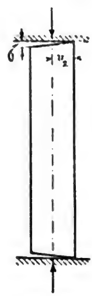


FIG. 22.

The deflection of the column may be divided into three stages. At first it will deflect as a position-fixed eccentrically loaded column (eccentricity $= v_2$) until the end cross sections bear on the abutments. It will then continue to deflect, but become a direction-fixed column. When the load reaches a certain value, the end cross sections will "swing round," and it will again become a position-fixed eccentrically loaded column.

To the first stage the formulæ of Case I, Variation 3, will apply. Substituting $v_2$ for $\epsilon_2$, equations (42), (43), and (46) become, respectively,

$$y = v_2 \sec \frac{aL}{2} \cos ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

$$y_0 = v_2 \sec \frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

$$f_c = \frac{W}{a}\left(1 + \frac{v_2^2}{\kappa^2} \sec \frac{aL}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

At the moment when the end cross sections bear on the abutments, the slope at the ends of the column will be $-\sigma$. Call the value of the load at

this moment $W_1$. Then $a_1^2 = \dfrac{W_1}{EI\left(1 - \dfrac{W_1}{Ea}\right)}$. Hence, from equation (208),

$\dfrac{dy}{dx} = -a_1 v_2 \sec\dfrac{a_1 L}{2}\sin a_1 x$ ; when $x = \dfrac{L}{2}$, $\dfrac{dy}{dx} = -\sigma$, and therefore

$a_1 v_2 \tan\dfrac{a_1 L}{2} = \sigma$. Consequently, $\tan\dfrac{a_1 L}{2} = \dfrac{\sigma}{a_1 v_2}$ . . . . . . . . . (211)

and $$\sec\frac{a_1 L}{2} = \sqrt{1 + \frac{\sigma^2}{a_1^2 v_2^2}}.$$

Hence equations (208) and (209) become, respectively,

$$y = \sqrt{v_2^2 + \frac{\sigma^2}{a_1^2}}\cos a_1 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

and $$y_0 = \sqrt{v_2^2 + \frac{\sigma^2}{a_1^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

The value of $W_1$ is obtained from equation (211), $\tan\dfrac{a_1 L}{2} = \dfrac{\sigma}{a_1 v_2}$, which may be written approximately

$$\tan\frac{\pi}{2}\sqrt{\frac{W_1}{P}} = \frac{\sigma}{v_2}\sqrt{\frac{EI}{W_1}} = \frac{\sigma L}{\pi v_2}\sqrt{\frac{P}{W_1}} \quad . \quad (214)$$

Equations (212), (213), and (214) apply only to the moment when the end cross sections come in contact with the abutments. After this the increase of load is spread uniformly over the end cross sections, and fixing moments $M_a$ are set up at the ends. The column has entered on stage two, during which it acts as a direction-fixed column with an original deflection. The original deflection anywhere, $y_1$, of the second stage is obviously the final deflection of the first stage. That is to say, from equation (212)

$$y_1 = \sqrt{v_2^2 + \frac{\sigma^2}{a_1^2}}\cos a_1 x \quad . \quad . \quad . \quad . \quad (215)$$

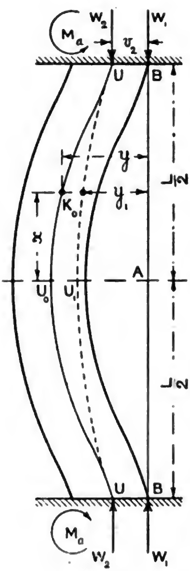


FIG. 23.

Take origin at A, Fig. 23, as in stage one, and let $W_2$ be the increase of load during stage two. Then $W_1 + W_2 = W$, the total load on the column.

Let $UU_1U$ be the position of the line of resistance of the column at the end of stage one, $UU_0U$ its position at some period during stage two. Then the bending moment at any point $K_0$ is

$$M = W_1 y + W_2(y - v_2) + M_a \quad . \quad (216)$$

and therefore the bending moment producing the change of curvature is

$$\begin{aligned} M - W_1 y_1 &= W_1 y + W_2(y - v_2) + M_a - W_1 y_1 \\ &= (W_1 + W_2)y + M_a - W_1 y_1 - W_2 v_2. \end{aligned}$$

Hence equation (105) becomes

$$\frac{d^2}{dx^2}(y - y_1) + \frac{(W_1 + W_2)y + M_a - W_1y_1 - W_2v_2}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)} = 0 \quad . \quad (217)$$

From equation (215)

$$\frac{d^2y_1}{dx^2} = -a_1\sqrt{a_1^2v_2^2 + \sigma^2} \cdot \cos a_1x.$$

Approximately also

$$\frac{W_1y_1}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)} = \frac{W_1y_1}{EI\left(1 - \frac{W_1}{Ea}\right)} = a_1^2y_1 = a_1\sqrt{a_1^2v_2^2 + \sigma^2} \cdot \cos a_1x.$$

Equation (217) becomes, therefore,

$$\frac{d^2y}{dx^2} + a^2y + \frac{a_2(M_a - W_2v_2)}{W_1 + W_2} = 0,$$

where

$$a^2 = \frac{W_1 + W_2}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)}.$$

To this the solution is

$$y + \frac{M_a - W_2v_2}{W_1 + W_2} = m \sin ax + n \cos ax.$$

When $x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$. Hence $y + \frac{M_a - W_2v_2}{W_1 + W_2} = n \cos ax$ and $\frac{dy}{dx} = -an \sin ax$.

When $x = \frac{L}{2}$, $\frac{dy}{dx} = -\sigma = -\sqrt{a_1^2v_2^2 + \sigma^2} \cdot \sin \frac{a_1L}{2}$.

Therefore $$n = \frac{\sigma}{a} \operatorname{cosec} \frac{aL}{2}.$$ When $x = \frac{L}{2}$, $y = v_2$.

Hence

$$v_2 + \frac{M_a - W_2v_2}{W_1 + W_2} = \frac{\sigma}{a} \cot \frac{aL}{2}$$

$$\frac{M_a - W_2v_2}{W_1 + W_2} = \frac{\sigma}{a} \cot \frac{aL}{2} - v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

Whence

$$y = \frac{\sigma}{a} \operatorname{cosec} \frac{aL}{2} \cos ax - \frac{\sigma}{a} \cot \frac{aL}{2} + v_2$$

$$= \frac{\sigma}{a}\left(\operatorname{cosec} \frac{aL}{2} \cos ax - \cot \frac{aL}{2}\right) + v_2 \quad . \quad . \quad . \quad . \quad (219)$$

and the maximum deflection at the centre

$$y_0 = \frac{\sigma}{a}\left(\operatorname{cosec} \frac{aL}{2} - \cot \frac{aL}{2}\right) + v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

These equations apply during the second stage.

The maximum bending moment at the centre [from equation (216)] is

$$M = (W_1 + W_2)y_0 - W_2v_2 + M_a.$$

Inserting the values of $y_0$ and $(W_2v_2 + M_a)$ from equations (220) and (218),

$$M = \frac{\sigma}{a}(W_1 + W_2)\operatorname{cosec}\frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221)$$

Equation (18), giving the maximum compressive stress at the centre of the column, becomes, therefore,

$$f_c = E\left\{v_2\frac{W_1 + W_2}{EI}\cdot\frac{\sigma}{a}\operatorname{cosec}\frac{aL}{2} + \frac{W_1 + W_2}{Ea}\right\}$$

$$= \frac{W_1 + W_2}{a}\left\{1 + \frac{v_2}{\kappa^2}\cdot\frac{\sigma}{a}\operatorname{cosec}\frac{aL}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

The bending moment at the ends of the column, from equation (216), is

$$M = (W_1 + W_2)v_2 - W_2v_2 + M_a.$$

Inserting the value of $(M_a - W_2v_2)$ from equation (218),

$$M = \frac{\sigma}{a}(W_1 + W_2)\cot\frac{aL}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (223)$$

Equation (18), giving the maximum compressive stress at the ends, becomes, therefore,

$$f_c = E\left\{\frac{W_1 + W_2}{Ea} - v_1\frac{W_1 + W_2}{EI}\frac{\sigma}{a}\cot\frac{aL}{2}\right\}$$

$$= \frac{W_1 + W_2}{a}\left\{1 - \frac{v_1}{\kappa^2}\cdot\frac{\sigma}{a}\cot\frac{aL}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (224)$$

In a similar way it may be shown that the maximum tensile stress at the ends of the column is

$$f_t = \frac{W_1 + W_2}{a}\left\{-\frac{v_2}{\kappa^2}\cdot\frac{\sigma}{a}\cot\frac{aL}{2} - 1\right\} \quad . \quad . \quad . \quad . \quad . \quad (225)$$

The third stage commences when $f_t = O$, i.e. when

$$\cot\frac{aL}{2} = -\frac{a\kappa^2}{\sigma v_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

From this equation the value of $a$, and hence that of the load at which "swinging round" occurs, might be determined. Provided that the material has not passed the elastic limit before or after "swinging round," the deflections and stresses in the column might be determined by an extension of the formulæ of Case I, Variation 6.

Introducing the same approximations as in previous cases, equation (222) may be written

$$f_c = \frac{W}{a}\left\{1 + \frac{v_2}{\kappa^2}\cdot\frac{\sigma L}{2\pi}\sqrt{\frac{P_2}{W}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}}\right\} \quad . \quad . \quad . \quad (227)$$

an expression for the maximum stress at the centre of the column during the second stage.

In a similar way, from equation (224),

$$f_c = \frac{W}{a}\left\{1 - \frac{v_1}{\kappa^2}\cdot\frac{\sigma L}{2\pi}\sqrt{\frac{P_2}{W}}\cot\pi\sqrt{\frac{W}{P_2}}\right\} \quad . \quad . \quad . \quad . \quad (228)$$

an expression for the maximum compressive stress at the ends.

The equations in the second stage of this Variation will be found to be identical with those of Case II, Variation 6, for an imperfectly fixed column, the end conditions being, in fact, identical. Like equation (157), equations

(224) and (228) call for comment. It would appear from the negative sign that the effect of the bending moment is to decrease the compressive stress on the layer of fibres $v = v_1$ up to the point where $\frac{aL}{2} = \pi\sqrt{\frac{W}{P_2}} = \frac{\pi}{2}$, and then to increase it. If, however, the expression for the bending moment at the end of the column, equation (223), be equated to zero,

$$\cot\frac{aL}{2} = 0,\ \frac{aL}{2} = \frac{\pi}{2},\ \text{or } W = \frac{\pi^2 EI}{L^2}.$$

That is to say, the total load on the column must reach a value $W = P = \frac{P_2}{4}$, equal to Euler's crippling load for the column if position-fixed, before direction-fixing becomes effective.

It would appear, in fact, that the second stage might again be divided into two. During the first stage the column is eccentrically loaded, and the load increases from zero to $W_1$. At this point the ends begin to bear all over their area and the second stage commences. The direction-fixing moment is now negative and equal to $W_1 v_2$. As the load increases, this negative moment decreases until, when $W_1 + W_2 = P = \frac{P_2}{4}$, it becomes zero, and the column is, in effect, position-fixed. The first part of the second stage ends here. When $W_1 + W_2$ exceeds P, the direction-fixing moment is positive, and the member acts as a direction-fixed column. This continues until the stress at the contour of the end cross sections becomes zero, when the second part of the second stage ends, the ends of the column "swing round," and the third stage commences.

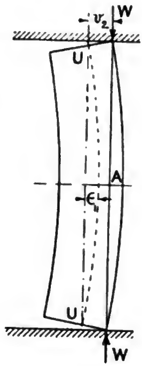


FIG. 24.

*Assumption* (*b*).—The column is of uniform cross section, but not originally straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The column is compressed between two perfectly flat surfaces which are always perpendicular to the line of action of the load. The ends of the column are perfectly flat surfaces which are perpendicular to the central axis (Fig. 24).

As in Assumption (*a*), the increase in the deflection of the column as the load increases may be divided into three stages. During the first it will deflect as a position-fixed column eccentrically loaded and originally curved. When the end cross sections bear on the abutments, the second stage will commence, and the column will become a position- and direction-fixed column. When the load reaches a certain value, the end cross sections will "swing round," and the specimen will again become a position-fixed eccentrically loaded column.

During stage one the formulæ of Case I, Variation 6, will apply, except that $\epsilon_1$, the original deflection, will be negative,* that is to say, the original deflection will tend to reduce the effect of the eccentricity. Let the equation to the original shape of the line of resistance be [see equation (82)]:

$$y_1 = v_2 - \epsilon_1\left\{1 - \frac{4x^2}{L^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (229)$$

* It is presumed, of course, that $\epsilon_1$ is less than $v_2$.

Then, from equation (84),

$$y = \left(v_2 - \frac{8\epsilon_1}{a^2L^2}\right) \sec \frac{aL}{2} \cos ax + \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad . \quad (230)$$

But when W reaches a certain value $W_1$, the end cross sections will commence to bear on the abutments; and $\frac{dy}{dx}$, when $x = \frac{L}{2}$, will equal zero. Let $a = a_1$ when $W = W_1$. Then $\frac{dy}{dx} = -a_1\left(v_2 - \frac{8\epsilon_1}{a_1^2L^2}\right) \sec \frac{a_1L}{2} \sin a_1x$, or, when $x = \frac{L}{2}$,

$$-a_1\left(v_2 - \frac{8\epsilon_1}{a_1^2L^2}\right) \tan \frac{a_1L}{2} = 0.$$

from which it follows that, unless $\tan \frac{a_1L}{2} = 0$, which cannot be the case,

$$v_2 = \frac{8\epsilon_1}{a_1^2L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231)$$

Substituting this value in equation (230)

$$y = v_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

That is to say, the column becomes straight.* The value of the load at which this occurs may be obtained from equation (231):

$$a_1^2 = \frac{8\epsilon_1}{v_2L^2}, \text{ or approximately } \frac{W_1}{EI} = \frac{8\epsilon_1}{v_2L^2}, \text{ from which}$$

$$W_1 = \frac{2\epsilon_1}{\pi^2 v_2} P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

When the second stage commences, therefore, the column has become straight. It acts, therefore, during this stage as an originally straight position- and direction-fixed column, and it may be shown as in Variation I that the least load under which flexure is possible is

$$W = W_1 + W_2 = \frac{4\pi^2EI}{L^2} = P_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

where $W_2$ is the increase in load during the second stage. The point of application of the load does not affect the result.

*Assumption* (*c*).—The above result is somewhat exceptional in that the column becomes straight. It will next be assumed that the original curvature was sinusoidal instead of parabolic, the other conditions remaining the same. Then the equation to the original shape of the line of resistance will be

$$y_1 = v_2 - \epsilon_1 \cos \frac{\pi x}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (235)$$

The differential equation giving its final shape is

$$\frac{d^2y}{dx^2} - \frac{\pi^2\epsilon_1}{L^2} \cos \frac{\pi x}{L} + \frac{Wy}{I\left(E - \frac{W}{a}\right)} = 0 \quad . \quad . \quad . \quad . \quad (236)$$

to which the solution is $y = m \sin ax + n \cos ax - \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2} \cos \frac{\pi x}{L}$. When

* It is probable that had the exact expression for the radius of curvature been used, this result would not have been obtained.

$x = 0$, $\frac{dy}{dx} = 0$, and $m = 0$. When $x = \frac{L}{2}$, $y = v_2$, and hence $n = v_2 \sec \frac{aL}{2}$.

Therefore,
$$y = v_2 \sec \frac{aL}{2} \cos ax - \frac{\pi^2 \epsilon_1}{\pi^2 - a^2L^2} \cos \frac{\pi x}{L} \quad . \quad . \quad . \quad . \quad (237)$$

where
$$a = \frac{W}{I\left(E - \frac{W}{a}\right)}$$

When W reaches a certain value $W_1$, the end cross sections will commence to bear on the abutments, and $\frac{dy}{dx}$, when $x = \frac{L}{2}$, will equal zero. Let $a = a_1$ when $W = W_1$. Then

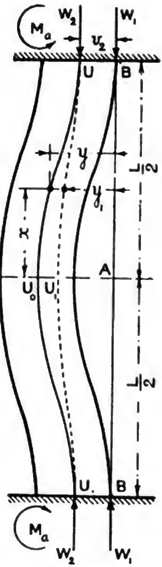


FIG. 25.

$$\frac{dy}{dx} = -a_1 v_2 \sec \frac{a_1 L}{2} \sin a_1 x + \frac{\pi}{L} \frac{\pi^2 \epsilon_1}{\pi^2 - a_1^2 L^2} \sin \frac{\pi x}{L},$$

or, when $x = \frac{L}{2}$,

$$0 = -a_1 v_2 \tan \frac{a_1 L}{2} + \frac{\pi}{L} \frac{\pi^2 \epsilon_1}{\pi^2 - a_1^2 L^2}, \text{ from which}$$

$$\tan \frac{a_1 L}{2} = \frac{\pi}{a_1 v_2 L} \cdot \frac{\pi^2 \epsilon_1}{\pi^2 - a_1^2 L^2} \quad . \quad . \quad (238)$$

the solution to which gives the value of $a_1$, and hence $W_1$. Equation (237) may therefore be written, when $a = a_1$,

$$y = v_2 \sec \frac{a_1 L}{2} \left\{ \cos a_1 x - \frac{a_1 L}{\pi} \sin \frac{a_1 L}{2} \cos \frac{\pi x}{L} \right\} \quad (239)$$

This is the value of the deflection at the end of the first stage, and therefore at the beginning of the second, for which stage $y$ in equation (239) becomes $y_1$.

Take origin at A, Fig. 25, as in stage one, and let $W_2$ be the increase of load during stage two. Then $W_1 + W_2 = W$ is the total load on the column.

Let $UU_1U$ be the position of the line of resistance at the end of stage one, $UU_0U$ its position at some period during stage two.

Then the bending moment anywhere is

$$M = W_1 y + W_2 (y - v_2) + M_a,$$

Hence the bending moment producing the change of curvature, $\frac{d^2}{dx^2}(y - y_1)$ is

$$(W_1 + W_2) y + M_a - W_2 v_2 - W_1 y_1.$$

Equation (105) becomes, therefore,

$$\frac{d^2}{dx^2}(y - y_1) + \frac{(W_1 + W_2) y + M_a - W_2 v_2 - W_1 y_1}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)} = 0 \quad . \quad . \quad . \quad (240)$$

From equation (239)

$$\frac{d^2 y_1}{dx^2} = v_2 \sec \frac{a_1 L}{2} \left\{ -a_1^2 \cos a_1 x + \frac{a_1 \pi}{L} \sin \frac{a_1 L}{2} \cos \frac{\pi x}{L} \right\}$$

and approximately

$$\frac{W_1 y_1}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)} = \frac{W_1 y_1}{EI\left(1 - \frac{W_1}{Ea}\right)} = a_1{}^2 y_1$$

$$= v_2 \sec\frac{a_1 L}{2}\left\{a_1{}^2 \cos a_1 x - \frac{a_1{}^3 L}{\pi} \sin\frac{a_1 L}{2} \cos\frac{\pi x}{L}\right\}.$$

Equation (240) becomes, therefore,

$$\frac{d^2y}{dx^2} - \frac{\pi^2\epsilon_1}{L^2}\cos\frac{\pi x}{L} + a^2 y + \frac{M_a - W_2 v_2}{W_1 + W_2} a^2 = 0 \quad . \quad . \quad . \quad (241)$$

where

$$\frac{W_1 + W_2}{EI\left(1 - \frac{W_1 + W_2}{Ea}\right)} = a^2.$$

The solution to this is

$$y + \frac{M_a - W_2 v_2}{W_1 + W_2} = m \sin ax + n \cos ax - \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2}\cos\frac{\pi x}{L}.$$

When $x = 0,\ \frac{dy}{dx} = 0$, and $m = 0$.

When $x = \frac{L}{2},\ \frac{dy}{dx} = 0$, and $n = \frac{\pi^3\epsilon_1}{aL(\pi^2 - a^2L^2)}\operatorname{cosec}\frac{aL}{2}$.

Hence, $$y + \frac{M_a - W_2 v_2}{W_1 + W_2} = \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2}\left\{\frac{\pi}{aL}\operatorname{cosec}\frac{aL}{2}\cos ax - \cos\frac{\pi x}{L}\right\} \quad . \quad (242)$$

When $x = \frac{L}{2},\ y = v_2$. Therefore,

$$\frac{M_a - W_2 v_2}{W_1 + W_2} = \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2}\left\{\frac{\pi}{aL}\cot\frac{aL}{2}\right\} - v_2 \quad . \quad . \quad . \quad (243)$$

Inserting this value in equation (242),

$$y = \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2}\left\{\frac{\pi}{aL}\left(\operatorname{cosec}\frac{aL}{2}\cos ax - \cot\frac{aL}{2}\right) - \cos\frac{\pi x}{L}\right\} + v_2 \quad . \quad (244)$$

and $$y_0 = \frac{\pi^2\epsilon_1}{\pi^2 - a^2L^2}\left\{\frac{\pi}{aL}\left(\operatorname{cosec}\frac{aL}{2} - \cot\frac{aL}{2}\right) - 1\right\} + v_2 \quad . \quad (245)$$

From these formulæ the bending moment everywhere, and hence the stress in the material, can be found.

## Variation 5. The Ordinary Column.

In the case of the ordinary column with flat ends it is necessary to take the effect of imperfect bearings into account in addition to the ordinary imperfections, which latter, as has been seen (p. 40), may be accounted for by assuming the column to be both eccentrically loaded and initially curved. As before, it will be assumed that all the imperfections tend to produce flexure in the plane perpendicular to the principal axis of elasticity about which S is a minimum. The bending will then be uniplanar.

From the nature of the case, however, as an inspection of Fig. 26 will show, it is impossible that all the imperfections will tend to produce flexure in the same direction. The largest eccentricity of loading will be set up by the initial curvature, and tend to produce flexure in the reverse direction to the

initial curvature, which reverse direction, in the later stages of the loading, becomes, or rather may become, the direction of the deflection. The initial deflection of the central axis must therefore be looked upon as negative. The effect of variations in the modulus of elasticity must be added algebraically to the initial deflection and eccentricity of loading, its true sign depending on whether it tends to produce deflection in the same direction as the initial curvature or the eccentricity.

The load line will not of necessity pass through the extreme corners of the specimen which touch the abutments, but in general will lie inside them, as shown in the figure. Further, the end cross sections will not, in general, be at right angles to the central axis or to the line of resistance, nor the abutments at right angles to the load line. The angle between the end cross sections and the abutments will determine the period during which the column will deflect without the ends bearing fully on the abutments.

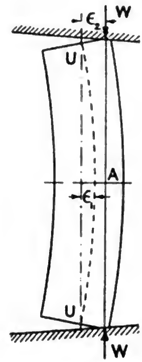


FIG. 26.

The deflection of the column must, in fact, as in the previous Variations, be divided into three stages. In the first, the specimen will act as a position-fixed eccentrically loaded and initially curved member. During the second, when the ends bear fully on the abutments, as a position- and direction-fixed column of which the direction-fixing is imperfect. At some point the end cross sections will rotate about their edges, when the third stage commences, and the column again becomes an eccentrically loaded position-fixed column.

To the first stage the formulæ of Case I, Variation 6, will apply, $\epsilon_1$ being negative. The equation to the initial shape of the line of resistance becomes, from equation (82),

$$y_1 = \epsilon_2 - \epsilon_1\left(1 - \frac{4x^2}{L^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (246)$$

Equation (84) gives the shape of the line of resistance under a load W

$$y = \left(\epsilon_2 - \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}\cos ax + \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad (247)$$

The maximum value of $y$ is

$$y_0 = \left(\epsilon_2 - \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad . \quad . \quad (248)$$

Now $$\epsilon_1 = \epsilon_3 \pm \epsilon_5 = \epsilon_3 \pm e_5\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

and $$\epsilon_2 = \epsilon_4 \pm \epsilon_6 = \epsilon_4 \pm e_6\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (250)$$

Where $\epsilon_1$ = the total original deflection of the line of resistance.
$\epsilon_3$ = the original deflection of the central axis.
$\epsilon_5$ = the original deflection of the line of resistance due to variations in the modulus of elasticity.
$\epsilon_2$ = the total eccentricity of the load.
$\epsilon_4$ = the eccentricity due to want of centering.
$\epsilon_6$ = the eccentricity due to variations in the modulus of elasticity.

See p. 42, Fig. 11, and equations (89) and (90).

The stresses in the material can be obtained from equations (91) and (94), (96) and (99), remembering, however, that $\epsilon_1$ is negative.

The first stage will continue until the end cross sections bear fully on the abutments, when the second stage will commence. At any point during the second stage, let $k\sigma$ be the value of $\frac{dy}{dx}$, the inclination of the line of resistance, at the ends of the column. Here $\sigma$ is a small angle depending in value on the original angle between the end cross sections and the abutments, and on the inclination of the ends of the line of resistance to the end cross sections. The coefficient $k$ expresses the angular yield of the abutments under the load.* Its value will be something greater than unity, and will vary with the load.

At the beginning of the second stage let $W = W_1$, $a = a_1$, and $k = k_1$, where

$$a_1^2 = \frac{W_1}{S\,(1 - s_a)}.$$

Then, from equation (247), when $x = \frac{L}{2}$,

$$\frac{dy}{dx} = -\,a_1\left(\epsilon_2 - \frac{8\epsilon_1}{a_1^2L^2}\right)\tan\frac{a_1L}{2} = -\,k_1\sigma$$

and
$$\cot\frac{a_1L}{2} = \frac{a_1}{k_1\sigma}\left(\epsilon_2 - \frac{8\epsilon_1}{a_1^2L^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (251)$$

by which equation $W_1$ is determined. The value of $y$ is given by equation (247), and is

$$y = \epsilon_2 + \frac{k_1\sigma}{a_1}\cot\frac{a_1L}{2}\left(\sec\frac{a_1L}{2}\cos a_1x - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (252)$$

For the second stage this deflection is the initial deflection $y_1$, and

$$\frac{d^2y_1}{dx^2} = -\,k_1\sigma a_1\,\mathrm{cosec}\,\frac{a_1L}{2}\cos a_1x.$$

Let $W_2$ be the increase of load during the second stage, so that the total load on the column during this stage is $W = W_1 + W_2$. Fig. 27 represents the condition of affairs. The column is acting as a position- and direction-fixed member, and $W_2$ may be supposed to act at a distance $\epsilon_7$ from U.

Take origin at A as in stage one, let $UU_1U$ be the position of the line of resistance at the end of stage one, and $UU_0U$ its position at some period during stage two.

Then the bending moment anywhere is

$$\begin{aligned} M &= W_1y + W_2\,(y - \epsilon_2 + \epsilon_7) + M_a \\ &= (W_1 + W_2)\,y + W_2\,(\epsilon_7 - \epsilon_2) + M_a \quad . \quad . \quad . \quad . \quad . \quad . \quad (253)\end{aligned}$$

* From some points of view it might be better to replace $k\sigma$ by some such expression as $(\sigma_1 + k\sigma_2)$, where $\sigma_1$ is a function of the original angle between the end cross section and the load line, and of that between the end cross section and the line of resistance; and $\sigma_2$ is a function of the angle between the load line and the abutment. The value of the angles in question cannot, however, be exactly determined, except, perhaps for an individual case, and the simpler form has been adopted in the text.

Hence the bending moment producing the change of curvature in the line of resistance is

$$(W_1 + W_2)y + W_2(\epsilon_7 - \epsilon_2) + M_a - W_1 y_1$$

and the differential equation to the line of resistance [equation (105)] becomes,

$$\frac{d^2}{dx^2}(y - y_1) + \frac{(W_1 + W_2)y + W_2(\epsilon_7 - \epsilon_2) + M_a - W_1 y_1}{S(1 - s_a)} = 0 \quad . \quad . \quad (254)$$

Now $\dfrac{W_1 y_1}{S(1 - s_a)} = a_1^2 y_1$, and from equation (252)

$$a_1^2 y_1 = a_1^2 \epsilon_2 + a_1 k_1 \sigma \operatorname{cosec} \frac{a_1 L}{2} \cos a_1 x - a_1 k_1 \sigma \cot \frac{a_1 L}{2}.$$

Further, from equation (251),

$$a_1 k_1 \sigma \cot \frac{a_1 L}{2} = a_1^2 \epsilon_2 - \frac{8\epsilon_1}{L^2}.$$

In virtue of these equations, equation (254) becomes

$$\frac{d^2 y}{dx^2} - \frac{8\epsilon_1}{L^2} + \frac{(W_1 + W_2)y + W_2(\epsilon_7 - \epsilon_2) + M_a}{S(1 - s_a)} = 0.$$

Let $a^2 = \dfrac{W_1 + W_2}{S(1 - s_a)}$. Then the solution to the differential equation is

$$y + \frac{W_2(\epsilon_7 - \epsilon_2) + M_a}{W_1 + W_2} = m \sin ax + n \cos ax + \frac{8\epsilon_1}{a^2 L^2}.$$

When $x = 0$, $\dfrac{dy}{dx} = 0$, and $m = 0$.

Hence $$\frac{dy}{dx} = -an \sin ax.$$

But when $x = \dfrac{L}{2}$, $\dfrac{dy}{dx} = -k\sigma$, therefore

$$n = \frac{k\sigma}{a} \operatorname{cosec} \frac{aL}{2}$$

and $$y + \frac{W_2(\epsilon_7 - \epsilon_2) + M_a}{W_1 + W_2} = \frac{8\epsilon_1}{a^2 L^2} + \frac{k\sigma}{a} \operatorname{cosec} \frac{aL}{2} \cos ax.$$

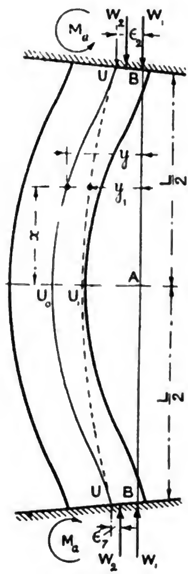


FIG. 27.

Further, when $x = \dfrac{L}{2}$, $y = \epsilon_2$, and

$$\frac{W_2(\epsilon_7 - \epsilon_2) + M_a}{W_1 + W_2} = \frac{8\epsilon_1}{a^2 L^2} + \frac{k\sigma}{a} \cot \frac{aL}{2} - \epsilon_2 \quad . \quad . \quad . \quad (255)$$

Whence $$y = \frac{k\sigma}{a}\left\{\operatorname{cosec} \frac{aL}{2} \cos ax - \cot \frac{aL}{2}\right\} + \epsilon_2 \quad . \quad . \quad . \quad (256)$$

and the maximum deflection at the centre of the column is

$$y_0 = \frac{k\sigma}{a}\left\{\operatorname{cosec}\frac{aL}{2} - \cot\frac{aL}{2}\right\} + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

The maximum bending moment at the centre, from equation (253), is

$$M = (W_1 + W_2)\,y_0 + W_2(\epsilon_7 - \epsilon_2) + M_a.$$

Inserting the values of $M_a$ and $y_0$ from equations (255) and (257),

$$M = (W_1 + W_2)\left\{\frac{k\sigma}{a}\operatorname{cosec}\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (258)$$

The maximum compressive stress at the centre on the concave side of the column, from equation (18), is

$$f_c = -E_2\left(u_2\frac{M}{S} + s_a\right).$$

Neglecting the negative sign denoting compression, and putting

$$u_2 = v_2 + \epsilon_5 + \epsilon_6,$$

$$f_c = E_2\left[(v_2 + \epsilon_5 + \epsilon_6)\frac{M}{S} + s_a\right],$$

which, by equation (258), becomes

$$f_c = E_2\left[(v_2 + \epsilon_5 + \epsilon_6)\frac{W_1 + W_2}{S}\left\{\frac{k\sigma}{a}\operatorname{cosec}\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} + s_a\right] \quad . \quad . \quad (259)$$

Making the same suppositions and approximations as in Case II, Variation 7 (p. 67), this reduces to

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{1}{\kappa^2}\left\{\frac{k\sigma}{a}\operatorname{cosec}\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\}\right] \quad . \quad . \quad (260)$$

Introducing the approximation $\frac{aL}{2} = \pi\sqrt{\frac{W}{P_2}}$, and neglecting the two factors containing $e$, as in previous cases, equation (260) becomes

$$f_c = \frac{W}{a}\left[1 + \frac{v_2L}{2\kappa^2}\cdot\frac{P_2}{\pi^2W}\left\{k\sigma\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} + \frac{4\epsilon_1}{L}\right\}\right] \quad . \quad . \quad (261)$$

which may be written

$$f_c = \frac{W}{a}\left[1 + \frac{v_2L}{2\kappa^2}\cdot k\sigma\left\{\frac{P_2}{\pi^2W}\left(\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right) + \frac{P_2}{\pi^2W}\left(1 + \frac{4\epsilon_1}{k\sigma L}\right)\right\}\right]$$

If $\frac{W}{P_2}$ be less than $\frac{1}{4}$, the approximation suggested in equation (178) may be used, when the expression becomes

$$f_c = \frac{W}{a}\left[1 + \frac{v_2L}{2\kappa^2}\cdot k\sigma\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2W}\left(1 + \frac{4\epsilon_1}{k\sigma L}\right)\right\}\right] \quad . \quad . \quad (262)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_c = \frac{W}{a}\left[1 + \frac{v_2}{\kappa^2}\left(\epsilon_1 + 0{\cdot}36\,k\frac{\sigma}{L}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (263)$$

It, however, by no means follows that the second stage will even have started when $\frac{W}{P_2}$ is as small as $\frac{1}{5}$.

The maximum bending moment at the ends of the column, from equation (253), is

$$M = (W_1 + W_2)\,\epsilon_2 + W_2\,(\epsilon_7 - \epsilon_2) + M_a.$$

Inserting the value of $M_a$ from equation (255),

$$M = (W_1 + W_2)\left\{\frac{k\sigma}{a}\cot\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} \quad . \quad . \quad . \quad . \qquad (264)$$

The maximum compressive stress at the ends of the column is obtained from equation (16)

$$f = E\left\{\frac{Mu}{S} - s_a\right\}$$

Now on the concave side of the column at the ends $u = u_1 = v_1 - \epsilon_6$, suppose the value of E to be $E_1'$.

Then $$f_c = -E_1'\left[-(v_1 - \epsilon_6)\frac{W_1 + W_2}{S}\left\{\frac{k\sigma}{a}\cot\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} + s_a\right]. \qquad (265)$$

Neglecting the negative sign of compression and making the same approximations as in Case II, Variation 7 (p. 68), this equation becomes

$$f_c = \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[1 - \left(v_1 - e_6\frac{a_1\bar{v}_1}{a}\right)\frac{1}{\kappa^2}\left\{\frac{k\sigma}{a}\cot\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\}\right] \quad . \qquad (266)$$

Here $E'_1 = E_a\left(1 + \frac{e_6}{2}\right)$ approximately, and $\epsilon_6 = e_6\frac{a_1\bar{v}_1}{a}$. This equation may be simplified in the manner adopted to simplify equation (260). It then becomes

$$f_c = \frac{W}{a}\left[1 - \frac{v_1L}{2\kappa^2}\cdot\frac{P_2}{\pi^2W}\left\{k\sigma\,\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}} + \frac{4\epsilon_1}{L}\right\}\right] \quad . \qquad (267)$$

Now, since at the time when the second stage begins the moment at the ends $W_1\epsilon_2$ is a negative direction-fixing moment, it follows that before the direction-fixing moment becomes positive the moment at the ends must have passed through the value zero. In this state the column would have been in the condition of a bent position-fixed column, and therefore the value of W must have exceeded $P = \frac{P_2}{4}$. Hence it follows that if the above equations represent the stress in the column in its position- and direction-fixed condition, $W > \frac{P_2}{4}$, and the approximate straight lines hitherto used as substitutes for the trigonometrical factor cannot be used.

The maximum stress on the convex side at the ends of the column is,* from equation (16),

$$f = E\left\{\frac{Mu}{S} - s_a\right\}$$

* No tensile stress can exist on the end cross sections, but for uniformity in the equations the stress on the convex side will be called $f_t$ as before.

On the convex side of the column at the ends $u = -u_2 = -(v_2 + \epsilon_6)$, and $E = E_2' = E_a\left(1 - \frac{e_6}{2}\right)$. Hence,

$$f_t = E_2'\left[(v_2 + \epsilon_6)\frac{W_1 + W_2}{S}\left\{\frac{k\sigma}{a}\cot\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} - s_a\right] \quad . \quad . \quad (268)$$

This, as before, may be written

$$f_t = \frac{W}{a}\left(1 - \frac{e_6}{2}\right)\left[\left(v_2 + e_6\frac{a_1\bar{v}_1}{a}\right)\frac{1}{\kappa^2}\left\{\frac{k\sigma}{a}\cot\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} - s_a\right]. \quad (269)$$

or approximately,

$$f_t = \frac{W}{a}\left[\frac{v_2L}{2\kappa^2}\cdot\frac{P_2}{\pi^2W}\left\{k\sigma\,\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}} + \frac{4\epsilon_1}{L}\right\} - 1\right] \quad . \quad (270)$$

Similar expressions for the maximum tensile stress at the centre of the columns are

$$f_t = \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[\left(v_1 - e_6\frac{a_1\bar{v}_1}{a}\right)\frac{1}{\kappa^2}\left\{\frac{k\sigma}{a}\operatorname{cosec}\frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2}\right\} - 1\right]. \quad (271)$$

or approximately,

$$f_t = \frac{W}{a}\left[\frac{v_1L}{2\kappa^2}\cdot\frac{P_2}{\pi^2W}\left\{k\sigma\,\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} + \frac{8\epsilon_1}{a^2L^2}\right\} - 1\right] \quad . \quad (272)$$

If $\frac{W}{P_2}$ be less than $\frac{1}{4}$,

$$f_t = \frac{W}{a}\left[\frac{v_1L}{2\kappa^2}k\sigma\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2W}\left(1 + \frac{4\epsilon_1}{k\sigma L}\right)\right\} - 1\right] \quad . \quad . \quad (273)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$f_t = \frac{W}{a}\left[\frac{v_2}{\kappa^2}\left(\epsilon_1 + 0{\cdot}36\,k\frac{\sigma}{L}\right) - 1\right]. \quad . \quad . \quad . \quad . \quad . \quad (274)$$

The above equations apply to the second stage. The third stage begins when the stress on the convex side of the end cross sections becomes zero. This stress is given by equation (270), and if $f_t =$ zero,

$$k\sigma\,\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}} = \frac{2\pi^2\kappa^2W}{v_2LP_2} - \frac{4\epsilon_1}{L} \quad . \quad . \quad . \quad . \quad (275)$$

This equation determines the load W at which the end sections will swing round on one edge.

**CASE IV. Columns with both Ends Fixed in Position, one End Fixed in Direction. Uniplanar Bending**

Suppose the upper end of the column to be fixed in position and the lower end to be fixed in both position and direction. Let $UU_0U$, Fig. 28, be the line of resistance of the bent column, $K_0$ any point thereon. Let A and B be the points of application of the load. Take origin at A, and let the

co-ordinates of $K_0$ be $x$ and $y$. The fixing moment at the upper end of the column is zero. Let $M_a$ be the fixing moment at the lower end. Then the horizontal forces FF called into play by the moment $M_a$ will be $\frac{M_a}{L}$. The actual bending moment at the point $K_0$ will be

$$M = Wy + M_a\left(1 - \frac{x}{L}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

Equation (103), giving the shape of the bent line of resistance, will become

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + M_a\left(1 - \frac{x}{L}\right)}{S\,(1 - s_a)} = 0 \quad . \quad (277)$$

The stress anywhere, in terms of the bending moment, from equation (104) is

$$f = E\left[\frac{u}{S}\left\{Wy + M_a\left(1 - \frac{x}{L}\right)\right\} - s_a\right] \quad (278)$$

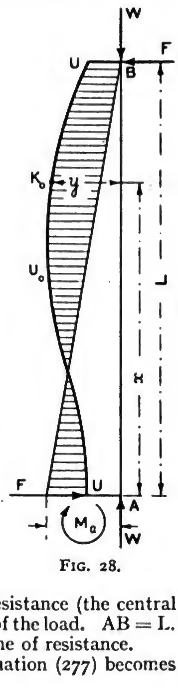



FIG. 28.

### VARIATION 1. IDEAL CONDITIONS

The column is of uniform cross section, and originally perfectly straight. The modulus of elasticity is constant everywhere, and the column is perfectly homogeneous. The load is applied at the centre of area of the end cross sections, and in the direction of the unstrained central axis. Suppose the column to bend.

Since the modulus of elasticity is constant, $s_a = \frac{W}{Ea}$.

The centre of resistance will coincide with the centre of area of the cross section, and the central axis will be the line of resistance. The moment of stiffness S will be constant and equal to EI, where I is the least moment of inertia of the cross section. The initial curvature $\frac{1}{\rho_1}$ will be zero.

Let $UU_0U$, Fig. 29, be the shape of the bent line of resistance (the central axis of the column). Take origin at A in the line of action of the load. AB = L. Let $x$ and $y$ be the co-ordinates of any point $K_0$ in the line of resistance.

Since the line of resistance was originally straight, equation (277) becomes

$$\frac{d^2y}{dx^2} + \frac{Wy + M_a\left(1 - \frac{x}{L}\right)}{EI\left(1 - \frac{W}{Ea}\right)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

Let $\frac{W}{I\left(E - \frac{W}{a}\right)} = a^2$. Then $\frac{d^2y}{dx^2} + a^2y + \frac{a^2M_a}{W} - \frac{a^2M_a}{W} \cdot \frac{x}{L} = 0,$

to which the solution is $\quad y + \frac{M_a}{W} = \frac{M_a}{WL}x + m \sin \alpha x + n \cos \alpha x.$

Hence $$\frac{dy}{dx} = \frac{M_a}{WL} + \alpha m \cos \alpha x - \alpha n \sin \alpha x.$$

When $x = 0, \frac{dy}{dx} = 0$, and $m = -\frac{M_a}{\alpha WL}$.

When $x = 0$, $y = 0$, and therefore $n = \frac{M_a}{W}$

Hence, $$y + \frac{M_a}{W} = \frac{M_a}{WL}x - \frac{M_a}{\alpha WL}\sin \alpha x + \frac{M_a}{W}\cos \alpha x \quad . \quad . \quad . \quad (280)$$

But when $x = L$, $y = 0$. Hence, if the column bend and $\frac{M_a}{W}$ have a value,

$$\alpha L = \tan \alpha L,$$

the solution to which is $\alpha L = 4{\cdot}493$.

But $$a^2 = \frac{W}{I\left(E - \frac{W}{a}\right)} = \left(\frac{4{\cdot}493}{L}\right)^2 = \frac{2{\cdot}047\,\pi^2}{L^2}$$

Hence, $$W = \frac{2{\cdot}047\,\frac{\pi^2 EI}{L^2}}{1 + 2{\cdot}047\,\frac{\pi^2 I}{aL^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (281)$$

If $\frac{W}{a}$ be neglected in comparison with E,

$$W = 2{\cdot}047\,\frac{\pi^2 EI}{L^2},$$

or approximately $$W = \frac{2\pi^2 EI}{L^2} = P_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (282)$$

### Variation 2. The Ordinary Column

In addition to the imperfections found in all ordinary columns, which, as has been seen, are equivalent to an initial curvature, together with an eccentricity of loading, there is, in the case of the ordinary column position-fixed at its upper end and position- and direction-fixed at its lower end, the effect of imperfect direction-fixing at the lower end to be taken into account. It will be assumed that the imperfection in the direction-fixing causes a definite increase in the slope at the lower end.

As in previous cases, it will be supposed that all the imperfections tend to produce flexure in the plane perpendicular to the principal axis of elasticity about which S is a minimum. The bending will then be uniplanar.

The column is assumed to be of uniform cross section. Let $VV_1V$, Fig. 30, be the original shape of the central axis, AB the line of action of the load, and $UU_1U$ the original shape of the line of resistance. Then UB is the eccentricity of the load $= \epsilon_2$. Of this eccentricity, VB $= \epsilon_4$ is due to inaccurate centering, and UV $= \epsilon_6$ to variations in the modulus of elasticity, $\epsilon_2 = \epsilon_4 + \epsilon_6$. $U_1U'$

is the original deflection of the line of resistance $= \epsilon_1$. Of this deflection, $U'U'' = V_1V' = \epsilon_3$ is due to the original deflection of the central axis, and $U_1U'' = \epsilon_5$ is the original deflection due to variations in the modulus of elasticity, $\epsilon_1 = \epsilon_3 + \epsilon_5$.

It will be assumed that the curve $UU_1U$ is a smooth plane curve and an arc of a parabola, its exact shape not being of great importance. Let $UU_0U$ be the final shape of the line of resistance. Take origin at A. AB = L.

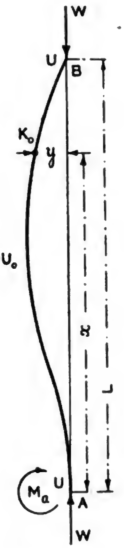



FIG. 29.

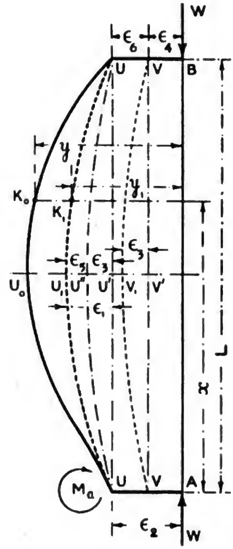



FIG. 30.

Let $x$ and $y$ be the co-ordinates of any point $K_0$ on the line of resistance in its final position, and suppose that $K_1$ was the original position of this point, and $x$ and $y_1$ its co-ordinates.

Then the equation to the line $UU_1U$ is

$$y_1 = \epsilon_2 + \epsilon_1\left(\frac{4x}{L} - \frac{4x^2}{L^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (283)$$

Hence $\dfrac{dy_1}{dx} = \epsilon_1\left\{\dfrac{4}{L} - \dfrac{8x}{L^2}\right\}$ and $\dfrac{d^2y_1}{dx^2} = -\dfrac{8\epsilon_1}{L^2}$. When $x = 0$, $\dfrac{dy_1}{dx} = +\dfrac{4\epsilon_1}{L}$.

Equation (277), giving the shape of the bent line of resistance, becomes for this case

$$\frac{d^2}{dx^2}(y - y_1) + \frac{Wy + \left(1 - \frac{x}{L}\right)M_a}{S(1 - s_a)} = 0,$$

or
$$\frac{d^2y}{dx^2}+\frac{8\epsilon_1}{L^2}+\frac{Wy+\left(1-\frac{x}{L}\right)M_a}{S\,(1-s_a)}=0 \quad . \quad . \quad . \quad . \qquad (284)$$

As before, S and $s_a$ will be assumed to be constant. Let $\alpha^2=\dfrac{W}{S\,(1-s_a)}$. Then the solution to the differential equation is

$$y=m\sin\alpha x+n\cos\alpha x-\frac{8\epsilon_1}{\alpha^2L^2}-\frac{M_a}{W}+\frac{M_a}{WL}x.$$

But when $x=0$, $y=\epsilon_2$. Hence, $n=\dfrac{8\epsilon_1}{\alpha^2L^2}+\dfrac{M_a}{W}+\epsilon_2$.

Now the lower end of the column is fixed both in position and direction, but the direction-fixing is imperfect. The original slope at the ends was $+\dfrac{4\epsilon_1}{L}$. Suppose that at the lower end, where $x=0$, the slope increases to $+\dfrac{4k\epsilon_1}{L}$, where $k$ is a coefficient greater than unity. The value of $k$ will probably vary with the magnitude of the load, but for the purposes of this analysis, $k$ is a constant. Then, when $x=0$, $\dfrac{dy}{dx}=\dfrac{4k\epsilon_1}{L}$, and $m=\dfrac{4k\epsilon_1}{\alpha L}-\dfrac{M_a}{\alpha LW}$. Hence, inserting the values of $m$ and $n$,

$$y=\left(\frac{4k\epsilon_1}{\alpha L}-\frac{M_a}{\alpha LW}\right)\sin\alpha x+\left(\frac{8\epsilon_1}{\alpha^2L^2}+\frac{M_a}{W}+\epsilon_2\right)\cos\alpha x-\frac{8\epsilon_1}{\alpha^2L^2}-\frac{M_a}{W}\left(1-\frac{x}{L}\right).$$

But when $x=L$, $y=\epsilon_2$. Therefore,

$$\frac{M_a}{W}=\frac{\dfrac{4k\epsilon_1}{\alpha L}\sin\alpha L+\left(\dfrac{8\epsilon_1}{\alpha^2L^2}+\epsilon_2\right)(\cos\alpha L-1)}{\dfrac{\sin\alpha L}{\alpha L}-\cos\alpha L} \quad . \quad . \quad . \quad . \qquad (285)$$

and the value of $y$ may be written

$$y=\frac{4k\epsilon_1}{\alpha L}\sin\alpha x+\left(\frac{8\epsilon_1}{\alpha^2L^2}+\epsilon_2\right)\cos\alpha x-\frac{8\epsilon_1}{\alpha^2L^2}$$
$$+\frac{M_a}{W}\left\{\cos\alpha x\quad\frac{\sin\alpha x}{\alpha L}+\frac{x}{L}-1\right\} \quad . \quad . \quad . \quad . \quad . \qquad (286)$$

The bending moment anywhere is

$$M=Wy+M_a\left(1-\frac{x}{L}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (287)$$

From equations (285), (286), and (287), the deflection and bending moment at every point in the length of the column can be found, and hence the stresses in the material. The position of the maximum deflection can be found by differentiating equation (286), but the value obtained for $x$ is not a simple expression.

## CHAPTER II

### *LATTICE-BRACED COLUMNS*

In the preceding chapter it has been assumed that the column acts as a homogeneous whole, that is to say, that no deformation of any sort is set up otherwise than the general flexure and compression of the column.

To built-up specimens and to specimens with thin walls this analysis will not apply directly. In them the local flexure and deformation, consequent upon the general flexure and compression, is the important factor which determines their resistance.

The majority of the large columns used in practice are of the built-up lattice-braced type, that is to say, they consist of two or more flanges united by braced webs. Experiment has shown that well-designed members of this type invariably fail owing to the flange buckling between the panel points, or to the flange plates buckling between the rivets, and it is necessary to modify the analysis to include this effect.

With the exception of No. 5, the whole of the assumptions made at the commencement of Chapter I will apply to the following analysis.

### The Lattice-Braced Column. Uniplanar Bending

In a lattice-braced column such as that shown in Fig. 31, suppose bending to take place on one plane only (uniplanar bending).

If the plane of bending be perpendicular to the lattice bracing, each flange will carry one-half the load, and may be treated as a separate column carrying a load $\frac{W}{2}$ and deflecting in the direction indicated. Provided that the flange be a solid section, the formulæ for solid columns will apply to this case.

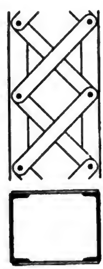


Fig. 31.

If, on the other hand, the column deflect in a plane parallel to the lattice bracing, an entirely new set of conditions arises to which, as has been stated, the laws for solid columns will not directly apply.

Suppose, then, that the column deflect in a plane parallel to the lattice bracing, and let $H_1H_2K_2K_1$, Fig. 32, be the original shape of one panel of the column. Let $J_1J_2K_2K_1$ be its shape when under strain, and $H_0K_0$ and $J_0K_0$ the unstrained and strained lines of resistance respectively. It is assumed that the length of the panel is short relative to the length of the column.

Let $O_1$ be the point of intersection of the lines $H_1H_2$, $K_1K_2$, and O the point of intersection of the lines $J_1J_2$, $K_1K_2$. Let $O_1K_0 = \rho_1$, $OK_0 = \rho$, the angle $H_0O_1K_0 = \theta_1$ and the angle $J_0OK_0 = \theta$. Now it will be presumed that both

the initial and final deflections of the column are very small; hence the angles $\theta_1$ and $\theta$ are also very small, and the distances $\rho_1$ and $\rho$ very very large. No sensible error will therefore be introduced by assuming the unstrained and strained lines of resistance to be very flat, smooth curves, of which $\rho_1$ and $\rho$ are the respective radii of curvature. Then, if $K_0K_1 = u_1$ and $K_0K_2 = u_2$,

$$K_1H_1 = (\rho_1 + u_1)\,\theta_1,\ K_1J_1 = (\rho + u_1)\,\theta,$$

$$K_2H_2 = (\rho_1 - u_2)\,\theta_1,\ K_2J_2 = (\rho - u_2)\,\theta,$$

and

$$(K_2H_2 - K_2J_2) = (\rho_1 - u_2)\,\theta_1 - (\rho - u_2)\,\theta.$$

But $(K_2H_2 - K_2J_2)$ is the contraction in length of $K_2H_2$, one of the series of elementary columns into which the flanges are divided by the lattice bracing. If the load-contraction curve of a column be examined, it will be found that, for the smaller values of the load, the curve is very very nearly a straight line,

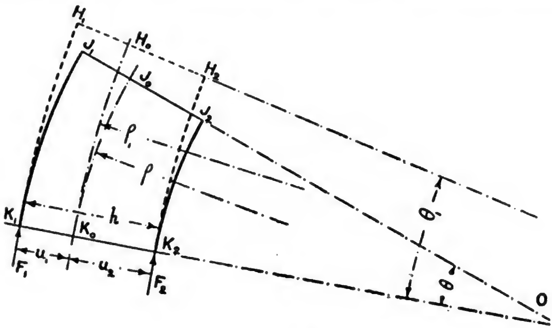



Fig. 32.

(see Fig. 33). In fact, the curve has similar characteristics to an ordinary stress-strain diagram, and within a certain limit, which might be called the proportional limit, the contraction in length $\delta L$ varies directly as the load varies, that is, $\delta L = cW$, where $c$ is a constant for the particular column. Further, it will be observed that the value of W, when the proportional limit is reached, is a considerable percentage of the ultimate strength. Assuming, therefore, that in practical cases a minimum factor of safety of four is prescribed, it follows that within the limits of practical working the contraction in length of the elementary columns into which the flanges are divided will be proportional to the loads on them.

Let $F_1$ be the load on the elementary column $K_1J_1$, and $c_1$ the constant for that column. Then $c_1F_1$ will be the contraction of length of that column. Similarly, if $F_2$ be the load on the elementary column $K_2J_2$ and $c_2$ the constant for that column, the contraction in length will be $c_2F_2$.

Now the forces in the flanges are made up of two parts: $F_a$, due to the direct compressive action of the load, and $F_b$, due to the bending moment. Let $F_a'$ be that part of the load on the elementary column $K_1J_1$ due to direct compressive action, and $F_a''$ the corresponding part in $K_2J_2$. Then

$$F_a' + F_a'' = W.$$

Further, the resultant of $F_a'$ and $F_a''$ will act along the line of resistance of the column. Hence, Fig. 34,

$$Wu_2 = F_a'(u_1 + u_2), \qquad F_a' = \frac{Wu_2}{u_1 + u_2}, \qquad F_a'' = \frac{Wu_1}{u_1 + u_2},$$

and the contraction in length of the two elementary columns due to these forces will be

$$c_1F_a' = \frac{Wc_1u_2}{u_1 + u_2} \text{ and } c_2F_a'' = \frac{Wc_2u_1}{u_1 + u_2}$$

respectively. But the contraction in length of the two elementary columns due simply to the direct compressive action of the load must be equal, hence

$$c_1F_a' = c_2F_a'' = \frac{Wc_1u_2}{u_1 + u_2} = \frac{Wc_2u_1}{u_1 + u_2}$$

and $$c_1u_2 = c_2u_1 \quad \text{. . . . . . . . . . (288)}$$

which determines the position of the line of resistance.

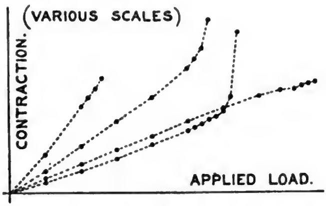



FIG. 33.—Load-Contraction Diagrams (various Experimenters).

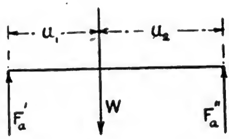



FIG. 34.

It has been shown above that the contraction of length of the elementary column $K_2H_2$ under the action of a force $F_2$ is

$$K_2H_2 - K_2J_2 = (\rho_1 - u_2)\,\theta_1 - (\rho - u_2)\,\theta = c_2F_2$$

$$= \theta_1\left\{(\rho_1 - u_2) - (\rho - u_2)\frac{\theta}{\theta_1}\right\} \quad \text{. . . . . (289)}$$

But $$\frac{\theta}{\theta_1} = \frac{K_0J_0}{K_0H_0} \cdot \frac{\rho_1}{\rho} = \left(1 - \frac{K_0H_0 - K_0J_0}{K_0H_0}\right)\frac{\rho_1}{\rho}$$

and $$K_0H_0 = \frac{K_2H_2 \times u_1 + K_1H_1 \times u_2}{u_1 + u_2},$$

$$K_0J_0 = \frac{K_2J_2 \times u_1 + K_1J_1 \times u_2}{u_1 + u_2}.$$

Hence $$\frac{\theta}{\theta_1} = \left\{1 - \frac{(K_2H_2 - K_2J_2)\,u_1 + (K_1H_1 - K_1J_1)\,u_2}{K_0H_0 \times (u_1 + u_2)}\right\}\frac{\rho_1}{\rho}.$$

But $(K_2H_2 - K_2J_2)$ is the contraction of the elementary column $K_2H_2$ under the load $F_2$, which contraction is equal to $c_2F_2$. Similarly, $(K_1H_1 - K_1J_1)$

is the contraction of the elementary column $K_1H_1$ under the load $F_1$, and is equal to $c_1F_1$. Let $K_0H_0$, the original length of the panel, equal $j$. Then

$$\frac{\theta}{\theta_1} = \left\{1 - \frac{c_2F_2u_1 + c_1F_1u_2}{j\,(u_1 + u_2)}\right\}\frac{\rho_1}{\rho},$$

or, since $F_1 + F_2 = W$, and by equation (288) $c_1u_2 = c_2u_1$,

$$\frac{\theta}{\theta_1} = \left\{1 - \frac{c_1u_2W}{j\,(u_1 + u_2)}\right\}\frac{\rho_1}{\rho}$$

Hence equation (289) becomes

$$c_2F_2 = \theta_1\left[(\rho_1 - u_2) - (\rho - u_2)\left\{1 - \frac{c_1u_2W}{j\,(u_1 + u_2)}\right\}\frac{\rho_1}{\rho}\right]$$

$$= \frac{j}{\rho\rho_1}\left[u_2\,(\rho_1 - \rho)\left\{1 - \frac{c_1u_2W}{j\,(u_1 + u_2)}\right\} + \rho\,(\rho_1 - u_2)\left\{\frac{c_1u_2W}{j\,(u_1 + u_2)}\right\}\right].$$

Now $u_2$ may be neglected in comparison with $\rho_1$, therefore

$$c_2F_2 = j\left[u_2\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{1 - \frac{c_1u_2W}{j\,(u_1 + u_2)}\right\} + \left\{\frac{c_1u_2W}{j\,(u_1 + u_2)}\right\}\right]$$

from which

$$ju_2\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{1 - \frac{c_1u_2W}{j\,(u_1 + u_2)}\right\} = \frac{c_2F_2\,(u_1 + u_2) - c_1u_2W}{(u_1 + u_2)}$$

$$\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = \frac{c_2F_2\,(u_1 + u_2) - c_1u_2W}{u_2\,\{j\,(u_1 + u_2) - c_1u_2W\}} \quad . \quad . \quad . \quad . \quad (290)$$

and

$$F_2 = \frac{\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)u_2\left\{j\,(u_1 + u_2) - c_1u_2W\right\} + c_1u_2W}{c_2\,(u_1 + u_2)} \quad . \quad . \quad (291)$$

which might be written

$$F_2 = \frac{\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{j\,(u_1 + u_2) - c_1u_2W\right\} + c_1W}{c_1 + c_2} \quad . \quad . \quad . \quad (292)$$

an expression for the force on the elementary column $K_2J_2$ in terms of the radii of curvature.

Now the moment of resistance of the column at any section $K_1K_0K_2$ may be obtained by taking moments about the point $K_0$,

$$M = F_2u_2 - F_1u_1,$$

or since $F_1 = W - F_2$

$$M = F_2\,(u_1 + u_2) - Wu_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

Substituting the value of $F_2$ from equation (291), the moment of resistance

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\frac{u_2}{c_2}\left\{j\,(u_1 + u_2) - c_1u_2W\right\} + \frac{c_1u_2W}{c_2} - Wu_1$$

this, by virtue of equation (288), may be written

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\frac{u_2}{c_2}\left\{j\,(u_1 + u_2) - c_2u_1W\right\} \quad . \quad . \quad . \quad (294)$$

or, if $u_1 + u_2 = h$,

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (295)$$

Now it was presupposed that both the initial and final curvatures were very small. Hence, if $x$ and $y_1$ be the co-ordinates of the point $K_0$ in its unstrained position,

$$\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) = -\frac{d^2}{dx^2}(y - y_1)$$

and

$$\frac{d^2}{dx^2}(y - y_1) + \frac{M}{\dfrac{h^2}{c_1 + c_2}\left\{j - \dfrac{Wc_1c_2}{c_1 + c_2}\right\}} = 0 \quad . \quad . \quad . \quad (296)$$

an equation strictly analogous to equation (15) obtained for solid columns

$$\frac{d^2}{dx^2}(y - y_1) + \frac{M}{S(1 - s_a)} = 0 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The solutions obtained to equation (15) may therefore be used for equation (296) if, for the denominator $S(1 - s_a)$, be substituted

$$\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}.$$

*The Value of c.*—It has been assumed in the above analysis that the alteration of length of one of the elementary columns into which the flange is divided by the bracing is a direct function of the load, in proof of which the shape of certain load-contraction diagrams has been adduced. It is necessary to enquire into the value of $c$ for a given length of column.

The contraction of a column of length L under a given load W is made up of two parts, (i) the shortening $\delta' L$ due to the direct compression of the load

$$\delta' L = L\frac{f_a}{E} = \frac{W}{Ea} L,$$

and (ii) the shortening $\delta'' L$ due to the deflection of the column. This latter is a complicated function of the length, the load, the original curvature and eccentricity. Examining the simplest of all cases, the originally straight, eccentrically loaded, position-fixed column, Case I, Variation 3, it will be seen from equations (42) and (43) that $y = y_0 \cos ax$

But

$$\int_0^l dl = \int_0^x \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \int_0^x \left\{1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\} dx.$$

Hence

$$l - x = \frac{a^2 y_0^2}{2}\int_0^x \sin^2 ax \cdot dx = \frac{a^2 y_0^2}{2}\left\{\frac{x}{2} - \frac{1}{4a}\sin 2ax\right\}.$$

The total shortening due to bending, therefore, is

$$\delta'' L = a^2 y_0 \left(\frac{L}{4} - \frac{1}{4a}\sin aL\right).$$

Inserting the value of $y_0$ from equation (43),

$$\delta''L = \frac{a^2L}{4} \epsilon_2^2 \sec^2 \frac{aL}{2} \left\{ 1 - \frac{2 \sin \frac{aL}{2} \cos \frac{aL}{2}}{aL} \right\}$$

$$= \frac{a^2\epsilon_2^2L}{4}\left\{1 - \frac{2}{aL} \tan \frac{aL}{2} + \tan^2 \frac{aL}{2}\right\}.$$

But $\frac{aL}{2} = \frac{\pi}{2}\sqrt{\frac{W}{P}}$ approximately. Hence

$$\delta''L = \frac{\pi^2\epsilon_2^2}{4L} \cdot \frac{W}{P}\left\{1 - \frac{2}{\pi}\sqrt{\frac{P}{W}} \tan \frac{\pi}{2}\sqrt{\frac{W}{P}} + \tan^2 \frac{\pi}{2}\sqrt{\frac{W}{P}}\right\}.$$

At the limit of working conditions $\frac{W}{P} = \frac{1}{4}$ and $\sqrt{\frac{W}{P}} = \frac{1}{2}$. In this case

$$\delta''L = \frac{\epsilon_2^2}{L} \cdot \frac{\pi}{8}(\pi - 2) = 0{\cdot}45 \frac{\epsilon_2^2}{L}.$$

To obtain an idea as to the magnitude of this contraction it will be assumed that $\epsilon_2 = 0{\cdot}001L$, whence it follows that $\delta''L = 0{\cdot}00000045L$. On the other hand, if $f_a = 5$ tons sq. in., and $E = 13{,}000$ tons sq. in.,

$$\delta'L = L \times \frac{5}{13{,}000} = 0{\cdot}00039L.$$

Hence $\delta L = \delta'L + \delta''L = 0{\cdot}00039045L$.

It is evident that the contraction $\delta''L$, due to deflection, is negligible compared with that due to direct compression, hence the fact that the load-contraction diagram is a straight line under working conditions. Even if the eccentricity were ten times that assumed, the shortening due to deflection would only be about 1 per cent. of the total contraction at the limit of working conditions. It follows, therefore, that variations in the value of E will have a far greater effect on the value of $F_2$, the load in the flange, than local original curvature or local eccentricity, and the shortening of the elementary flange columns may quite safely be determined by the law $\delta L = cW$, or, for the elementary flange column, $\delta j = c_2F_2$.

If, then, $\delta''L$ be neglected as small compared with $\delta'L$, it follows that

$$c = \frac{\delta L}{W} = \frac{\delta' L}{W} = \frac{1}{E} \cdot \frac{L}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (297)$$

and

$$c_1 = \frac{j}{E_1a_1},\ c_2 = \frac{j}{E_2a_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (298)$$

Substituting these values in the expression

$$\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (299)$$

it becomes

$$\frac{h^2}{\frac{j}{E_1a_1} + \frac{j}{E_2a_2}}\left\{j - \frac{W\frac{j^2}{E_1E_2a_1a_2}}{\frac{j}{E_1a_1} + \frac{j}{E_2a_2}}\right\}$$

$$= \frac{E_1E_2a_1a_2h^2}{E_1a_1 + E_2a_2}\left\{1 - \frac{W}{E_1a_1 + E_2a_2}\right\} = \frac{E_1E_2a_1a_2h^2}{E_1a_1 + E_2a_2} \quad . \quad . \quad (300)$$

[compare equation (74)], for W may be neglected in comparison with $(E_1a_1 + E_2a_2)$.

If, as will in general be the case, the two flanges of the column are equal in area, $a_1 = a_2 = \frac{a}{2}$. Also $E_1E_2 = E_a^2$, and $E_1 + E_2 = 2E_a$ approximately.

Then $$\frac{E_1E_2a_1a_2h^2}{E_1a_1 + E_2a_2} = \frac{E_a^2ah^2}{2(E_1 + E_2)} = E_a\frac{ah^2}{4} = E_aI,$$

where $\frac{ah^2}{4} = I$ is the moment of inertia of the column as a whole. Now if

$$\alpha^2 = \frac{W}{\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

then $$\alpha^2 = \frac{W(E_1a_1 + E_2a_2)}{E_1E_2a_1a_2h^2} = \frac{W}{E_aI} \text{ approximately} \quad . \quad . \quad . \quad (302)$$

and $\frac{\alpha L}{2} = \frac{\pi}{2}\sqrt{\frac{W}{P}}$, as in the case of the solid column.

Another approximation for (299) follows from the above. This expression may be written

$$\frac{jh^2}{c_1 + c_2}\left\{1 - \frac{Wc_1c_2}{j(c_1 + c_2)}\right\}.$$

But $\frac{Wc_1c_2}{j(c_1 + c_2)}$ may be neglected in comparison with unity. Hence

$$\alpha^2 = \frac{W(c_1 + c_2)}{jh^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (303)$$

*Recapitulation.*—The contraction in length of a column, provided the load does not exceed working limits, is $\delta L = cW$, where

$$c = \frac{L}{Ea} \text{ approximately} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (297)$$

The position of the line of resistance is given by the equation

$$c_1u_2 = c_2u_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (288)$$

Hence $$u_1 = \frac{c_1h}{c_1 + c_2},\ u_2 = \frac{c_2h}{c_1 + c_2}.$$

The force in a flange is

$$F_2 = \frac{\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\left\{j(u_1 + u_2) - c_1u_2W\right\} + c_1W}{c_1 + c_2} \quad . \quad . \quad . \quad (292)$$

$$F_1 = \frac{Wu_2 - M}{u_1 + u_2},$$

$$F_2 = \frac{M + Wu_1}{u_1 + u_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (293)$$

The moment of resistance is

$$M = \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\frac{u_2}{c_2}\left\{j(u_1 + u_2) - c_2 u_1 W\right\}. \quad . \quad . \quad . \quad (294)$$

$$= \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right)\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}. \quad . \quad . \quad . \quad . \quad (295)$$

The shape of the line of resistance is determined by

$$\frac{d^2}{dx^2}(y - y_1) + \frac{M}{\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}} = 0 \quad . \quad . \quad . \quad (296)$$

which is analogous to equation (15) if for the denominator $S(1 - s_a)$ be substituted

$$\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}.$$

The constant $$a^2 = \frac{W}{\frac{h^2}{c_1 + c_2}\left\{j - \frac{Wc_1c_2}{c_1 + c_2}\right\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (301)$$

$$= \frac{W}{\frac{E_1E_2a_1a_2h^2}{E_1a_1 + E_2a_2}} = \frac{W}{E_aI} \text{ approximately} \quad . \quad . \quad . \quad . \quad (302)$$

Hence $$\frac{aL}{2} = \frac{\pi}{2}\sqrt{\frac{W}{P}} = \pi\sqrt{\frac{W}{P_2}} \text{ approximately.}$$

## CASE V. Position-fixed Lattice-braced Columns. Uniplanar Bending

*Both ends fixed in position, but free in direction*

### The Ordinary Column

The same imperfections will be found in lattice-braced columns as in solid columns. The central axis will have an initial curvature, the load will be eccentric, and the modulus of elasticity will vary both in the direction of the width and length of the columns; that is to say, the modulus of elasticity will be different in the two flanges, and will not be uniform over the length of each. In addition, the elementary columns which form the flanges of the lattice-braced columns will suffer from the imperfections of the ordinary solid column. The effect of the variations in the modulus of elasticity and the imperfections in the elementary columns will be that $c_1$ will not be equal to $c_2$, and hence the line of resistance will not coincide with the central axis. Not only so, but the values of $c_1$ and $c_2$ will be different in different panels of the same flange, and hence the line of resistance will not be a straight line. In short, the result of the variations in the modulus of elasticity and the imperfections in the elementary columns is in effect an initial curvature of the line of resistance together with an eccentricity of loading. This is exactly analogous to the effect of variations in the modulus of elasticity in a solid column.

In addition, the variations in $c_1$ and $c_2$ will have the effect of varying the value of

$$\frac{h^2}{c_1 + c_2}\left(j - \frac{Wc_1c_2}{c_1 + c_2}\right).$$

As in the case of similar variations in the value of $S(1 - s_a)$ in solid columns, it may be shown that the effect is so small that it may be neglected.

It will be assumed that all the imperfections tend to produce flexure in a plane parallel to the lattice bracing. Suppose the column to be of uniform cross section, and that the panel length $j$ is uniform and relatively small compared with L.

Let $\epsilon_1 = \epsilon_3 + \epsilon_5$ be the original deflection, of which $\epsilon_3$ is the original deflection of the central axis, and $\epsilon_5$ that due to variations in $c_1$ and $c_2$. Let $\epsilon_2 = \epsilon_4 + \epsilon_6$ be the eccentricity of loading, of which $\epsilon_4$ is due to inaccurate centering and $\epsilon_6$ to variations in $c_1$ and $c_2$.

It will be assumed, as in the case of the solid column, that the initial shape of the line of resistance $UU_1U$, Fig. 11, is a smooth plane curve and an arc of a parabola. Let $UU_0U$ be the final shape of the line of resistance. Take origin at A, $AB = \frac{L}{2}$. Let $x$ and $y$ be the co-ordinates of any point $K_0$ on the line of resistance in its final position, and let $K_1$ be the original position of this point and $x$ and $y_1$ its co-ordinates.

Then the equation to the line $UU_0U$ is

$$y_1 = \epsilon_2 + \epsilon_1\left(1 - \frac{4x^2}{L^2}\right).$$

Hence, from equation (296),

$$\frac{d^2y}{dx^2} + \frac{Wy}{\dfrac{h^2}{c_1 + c_2}\left\{j - \dfrac{Wc_1c_2}{c_1 + c_2}\right\}} + \frac{8\epsilon_1}{L^2} = 0 \quad . \quad . \quad . \quad . \quad (304)$$

Let

$$a^2 = \frac{W}{\dfrac{h^2}{c_1 + c_2}\left\{j - \dfrac{Wc_1c_2}{c_1 + c_2}\right\}},$$

which, for the reasons already given, will be assumed constant. Then equation (304) becomes

$$\frac{d^2y}{dx^2} + \frac{8\epsilon_1}{L^2} + a^2y = 0,$$

to which the solution is [see equations (84) and (85)]

$$y = \left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}\cos ax - \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad . \quad (305)$$

and the maximum deflection

$$y_0 = \left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2} \quad . \quad . \quad . \quad . \quad . \quad (306)$$

The maximum force in a flange occurs at the centre on the concave side. From equation (293)

$$F_c = \frac{Wy_0 + Wu_1}{u_1 + u_2} = \frac{W}{h}\left\{\left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2} - \frac{8\epsilon_1}{a^2L^2} + \frac{c_1h}{c_1 + c_2}\right\}. \quad (307)$$

But from equation (298),

$$c_1 = \frac{j}{E_1 a_1} \text{ and } c_2 = \frac{j}{E_2 a_2}.$$

Hence $$\frac{c_1 h}{c_1 + c_2} = \frac{E_2 a_2}{E_1 a_1 + E_2 a_2} h.$$

In general, $a_1 = a_2$, when

$$\frac{c_1 h}{c_1 + c_2} = \frac{E_2}{E_1 + E_2} h = \frac{E_2}{E_a} \cdot \frac{h}{2}.$$

If the variation in the moduli of elasticity tend to produce bending in the same direction as the eccentricity of loading and initial curvature, it follows that $E_1$ is greater than $E_2$, and therefore $E_a$ is greater than $E_2$. An error on the safe side will consequently be made if $\frac{E_2}{E_a}$ be put equal to unity. If also the approximate value for $\frac{aL}{2}$ be introduced, equation (307) becomes

$$F_c = \frac{W}{h}\left\{\frac{h}{2} + \epsilon_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \frac{8\epsilon_1 P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\} \quad . \quad (308)$$

[Compare equation (91).]

This equation only holds within working limits, i.e. while $\frac{W}{P}$ is less than $\frac{1}{4}$, and it has been shown (p. 43) that within these limits the two functions

$$\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} \text{ and } \frac{8P}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)$$

are very nearly equal, and that they differ very little from the straight line

$$\left(1 + \frac{3}{2}\frac{W}{P}\right).$$

For working loads, therefore, equation (308) may be written

$$F_c = \frac{W}{h}\left\{\frac{h}{2} + (\epsilon_1 + \epsilon_2)\left(1 + \frac{3W}{2P}\right)\right\} \quad . \quad (309)$$

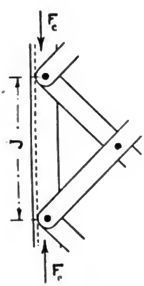


Fig. 35.

Inserting the value $\frac{W}{P} = \frac{1}{5}$ as an upper limit for working conditions, this expression becomes

$$F_c = W\left\{\frac{1}{2} + 1{\cdot}3\,\frac{\epsilon_1 + \epsilon_2}{h}\right\} \quad . \quad . \quad . \quad (310)$$

These formulæ determine the maximum load $F_c$ on one of the elementary flange columns (Fig. 35). It is next necessary to consider the stresses in this.

Let $\epsilon_1''$ and $\epsilon_2''$ be the local initial curvature and eccentricity of loading respectively. Then, provided the

ratio of $\frac{W}{P}$ for the elementary flange column be less than $\frac{1}{5}$, equation (93A) may be applied, which becomes

$$f_c = \frac{F_c}{a_2}\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad . \quad . \quad (311)$$

where $v_2''$ and $\kappa''$ have reference to the flange considered as a column between the panel points, and $P''$ is Euler's crippling load for the same elementary column. Hence, from equations (309) and (311),

$$f_c = \frac{2W}{a}\left\{\frac{1}{2} + \left(1 + \frac{3W}{2P}\right)\frac{\epsilon_1 + \epsilon_2}{h}\right\}\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad (312)$$

a formula for the maximum stress in the column. If the superior limit for $\frac{W}{P}$ and $\frac{F_c}{P''}$, namely $\frac{1}{5}$, be inserted in this equation, it reduces to

$$f_c = \frac{2W}{a}\left\{\frac{1}{2} + 1{\cdot}3\ \frac{\epsilon_1 + \epsilon_2}{h}\right\}\left\{1 + 1{\cdot}3\ \frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad (312A)$$

## CASE VI. Position- and Direction-fixed Lattice-braced Columns Uniplanar Bending

*Both ends fixed in position and direction*

### THE ORDINARY COLUMN

To the imperfections considered in the case of the position-fixed column, Case V, must be added in this case, the effect of imperfect direction-fixing. The column will therefore be assumed to be initially curved and eccentrically loaded. The imperfection in the direction-fixing will be allowed for by assuming a given increase in the slope of the ends of the column. It will be further assumed that all the imperfections tend to produce flexure in a plane parallel to the lattice bracing, that the column is of uniform cross section, and that the panel length $j$ is uniform and relatively small compared with L.

As before, let $\epsilon_1 = \epsilon_3 + \epsilon_5$ be the original deflection, of which $\epsilon_3$ is the original deflection of the central axis, and $\epsilon_5$ that due to variations in $c_1$ and $c_2$. Let $\epsilon_2 = \epsilon_4 + \epsilon_6$ be the eccentricity of loading, of which $\epsilon_4$ is due to inaccurate centering and $\epsilon_6$ to variations in $c_1$ and $c_2$.

It will be assumed that the initial shape of the line of resistance $UU_1U$, Fig. 19, is a smooth curve and an arc of a parabola. Let $UU_0U$ be the final shape of the line of resistance. Take origin at A, $AB = \frac{L}{2}$. Let $x$ and $y$ be the co-ordinates of any point $K_0$ on the line of resistance in its final position, and suppose that $K_1$ was the original position of this point and $x$ and $y_1$ its co-ordinates.

The equation to the line $UU_1U$ is

$$y_1 = \epsilon_2 + \epsilon_1\left(1 - \frac{4x^2}{L^2}\right)$$

Assuming that the column is symmetrical about the axis of $y$, $M_a = M_b$, and equation (296) becomes

$$\frac{d^2y}{dx^2} + \frac{Wy + M_a}{\dfrac{h^2}{c_1 + c_2}\left\{j - \dfrac{Wc_1c_2}{c_1 + c_2}\right\}} + \frac{8\epsilon_1}{L^2} = 0 \quad . \quad . \quad . \quad . \quad (313)$$

Let

$$a^2 = \frac{W}{\dfrac{h^2}{c_1 + c_2}\left\{j - \dfrac{Wc_1c_2}{c_1 + c_2}\right\}},$$

which, as before, will be assumed constant.

Then the solution to the differential equation is [see equation (165)]

$$y + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W} = \left(y_0 + \frac{8\epsilon_1}{a^2L^2} + \frac{M_a}{W}\right)\cos ax \quad . \quad . \quad . \quad (314)$$

The original slope of the line of resistance at the ends of the column where $x = \frac{L}{2}$ was $-\frac{4\epsilon_1}{L}$. Suppose, due to imperfect direction-fixing, that this increases to $-\frac{4k\epsilon_1}{L}$, where $k$ is a coefficient somewhat greater than unity. The value of $k$ will probably vary with the load, but for the purposes of this analysis $k$ is a constant. Then, since when $x = \frac{L}{2}$, $y = \epsilon_2$ and $\frac{dy}{dx} = -\frac{4k\epsilon_1}{L}$, it follows from equation (314) that

$$M_a = -\frac{8W\epsilon_1}{a^2L^2}\left\{1 - k \cdot \frac{aL}{2}\cot\frac{aL}{2}\right\} - W\epsilon_2 \quad . \quad . \quad . \quad (315)$$

$$y_0 = \frac{4k\epsilon_1}{aL}\left(\frac{1 - \cos\dfrac{aL}{2}}{\sin\dfrac{aL}{2}}\right) + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

and

$$y = \frac{4k\epsilon_1}{aL}\left(\frac{\cos ax - \cos\dfrac{aL}{2}}{\sin\dfrac{aL}{2}}\right) + \epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (317)$$

[Compare equations (166), (167), and (169), and see remarks thereon.]

The bending moment anywhere from equations (315) and (317) is

$$M = Wy + M_a = \frac{8W\epsilon_1}{a^2L^2}\left(k\frac{aL}{2}\cos ax \operatorname{cosec}\frac{aL}{2} - 1\right) \quad . \quad . \quad (318)$$

At the centre of the column where $x = 0$, this becomes

$$M_0 = \frac{8W\epsilon_1}{a^2L^2}\left(k\frac{aL}{2}\operatorname{cosec}\frac{aL}{2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (319)$$

But, from equation (293),

$$F_2 = \frac{M + Wu_1}{u_1 + u_2}.$$

Hence, from equations (319) and (288), $F_c$, the maximum load on an elementary flange column, at the centre is,

$$F_c = \frac{W}{h}\left[\frac{8\epsilon_1}{a^2L^2}\left(k\frac{aL}{2}\operatorname{cosec}\frac{aL}{2} - 1\right) + \frac{c_1 h}{c_1 + c_2}\right] \quad . \quad . \quad . \quad (320)$$

As in the case of the position-fixed column, it may be shown that $\frac{aL}{2} = \pi\sqrt{\frac{W}{P_2}}$ approximately, and that if $a_1 = a_2 = \frac{a}{2}$, $\frac{c_1 h}{c_1 + c_2} = \frac{E_2}{E_a}\cdot\frac{h}{2}$, which with safety may be taken as equal to $\frac{h}{2}$. Introducing these approximations into equation (320),

$$F_c = \frac{W}{h}\left[\frac{2P_2\epsilon_1}{\pi^2 W}\left(k\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right) + \frac{h}{2}\right] \quad . \quad . \quad (321)$$

This equation only holds within working limits $\left(\frac{W}{P_2} < \frac{1}{4}\right)$, and within these limits it has been shown (p. 68) that

$$\frac{1}{\pi^2}\frac{P_2}{W}\left\{\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1\right\} = 0{\cdot}17 + 0{\cdot}26\frac{W}{P_2}.$$

Equation (321) may be written, therefore,

$$F_c = \frac{W}{h}\left[\frac{h}{2} + \frac{2k\epsilon_1}{\pi^2}\frac{P_2}{W}\left\{\pi\sqrt{\frac{W}{P_2}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_2}} - 1 + 1 - \frac{1}{k}\right\}\right]$$

$$= \frac{W}{h}\left[\frac{h}{2} + 2k\epsilon_1\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2 W}\left(1 - \frac{1}{k}\right)\right\}\right] \quad . \quad . \quad . \quad (322)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$F_c = W\left[\frac{1}{2} + \frac{\epsilon_1}{h}(1{\cdot}44\,k - 1)\right] \quad . \quad . \quad . \quad . \quad . \quad (323)$$

If the direction-fixing be perfect, $k = 1$, and

$$F_c = W\left(\frac{1}{2} + 0{\cdot}44\frac{\epsilon_1}{h}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (324)$$

These formulæ determine the maximum load $F_c$ on one of the elementary flange columns at the centre of the main column. Knowing this load, the maximum stress in the material may be determined from equation (311)

$$f_c = \frac{F_c}{a_2}\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}$$

where $\epsilon_1''$, $\epsilon_2''$, $v_2''$, and $\kappa''$ have reference to the flange considered as a column between the panel points. From equations (311) and (322), therefore,

$$f_c = \frac{2W}{a}\left[\frac{1}{2} + \frac{2k\epsilon_1}{h}\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2 W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

$$\times\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad (325)$$

a formula for the maximum stress at the centre of the column. Giving to $\frac{W}{P_2}$ and $\frac{F_c}{P''}$ their superior limit $\frac{1}{5}$,

$$f_c = \frac{2W}{a}\left\{\frac{1}{2} + \frac{\epsilon_1}{h}(1 \cdot 44\,k - 1)\right\}\left\{1 + 1 \cdot 3\frac{v_2}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad (325\text{A})$$

At the ends of the column $x = \frac{L}{2}$, and the bending moment, from equation (318), is

$$M = \frac{8W\epsilon_1}{a^2L^2}\left(k\frac{aL}{2}\cot\frac{aL}{2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (326)$$

From equation (293) $F_1 = \frac{Wu_2 - M}{u_1 + u_2}$.

Hence, from equations (326) and (288), $F_c$, the maximum load on an elementary flange column, at the ends is

$$F_c = \frac{W}{h}\left[\frac{8\epsilon_1}{a^2L^2}\left(1 - k\frac{aL}{2}\cot\frac{aL}{2}\right) + \frac{c_2h}{c_1 + c_2}\right] \quad . \quad . \quad . \quad (327)$$

Introducing the same approximations as before, $\frac{aL}{2} = \pi\sqrt{\frac{W}{P_2}}$ approximately, and if $a_1 = a_2 = \frac{a}{2}$, $\frac{c_1h}{c_1 + c_2} = \frac{E_1}{E_a}\cdot\frac{h}{2}$, which will be somewhat greater than $\frac{h}{2}$. Equation (327) becomes, therefore,

$$F_c = \frac{W}{h}\left[\frac{E_1h}{2E_a} + \frac{2P_2\epsilon_1}{\pi^2W}\left\{1 - k\pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right\}\right] \quad . \quad . \quad (328)$$

This equation only holds within working limits $\left(\frac{W}{P_2} < \frac{1}{4}\right)$, and within these limits it has been shown (p. 68) that

$$\frac{1}{\pi^2}\cdot\frac{P_2}{W}\left\{1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right\} = 0 \cdot 33 + 0 \cdot 29\frac{W}{P_2}.$$

Equation (328) may be written

$$F_c = \frac{W}{h}\left[\frac{E_1h}{2E_a} + \frac{2k\epsilon_1}{\pi^2}\frac{P_2}{W}\left\{\frac{1}{k} - 1 + 1 - \pi\sqrt{\frac{W}{P_2}}\cot\pi\sqrt{\frac{W}{P_2}}\right\}\right]$$

$$= \frac{W}{h}\left[\frac{E_1h}{2E_a} + 2k\epsilon_1\left\{0 \cdot 33 + 0 \cdot 29\frac{W}{P_2} - \frac{1}{\pi^2}\cdot\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right] \quad . \quad (329)$$

Giving to $\frac{W}{P_2}$ its probable superior limit $\frac{1}{5}$,

$$F_c = W\left[\frac{E_1}{2E_a} + \frac{\epsilon_1}{h}\left\{1 - 0 \cdot 22\,k\right\}\right] \quad . \quad . \quad . \quad . \quad (330)$$

The value of $F_c$ diminishes as that of $k$ increases. It is safer, therefore, to give to $k$ its minimum value, unity. In this case the difference between

$E_1$ and $E_a$ may be neglected, and equations (329) and (330) become respectively

$$F_c = W\left[\frac{1}{2} + \frac{\epsilon_1}{h}\left\{0{\cdot}66 + 0{\cdot}58\,\frac{W}{P_2}\right\}\right] \quad . \quad . \quad . \quad . \quad (331)$$

and

$$F_c = W\left[\frac{1}{2} + 0{\cdot}78\,\frac{\epsilon_1}{h}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (332)$$

These formulæ determine the maximum load $F_c$ on one of the elementary flange columns at the ends of the main column. Knowing this load, the maximum stress in the material may be determined from equation (311)

$$f_c = \frac{F_c}{a_2}\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}$$

where $\epsilon_1''$, $\epsilon_2''$, $v_2''$, and $\kappa''$ have reference to the flange considered as a column between the panel points, and $P''$ is Euler's crippling load for the same elementary column. From equations (311) and (331), therefore, assuming that $k = 1$,

$$f_c = \frac{2W}{a}\left[\frac{1}{2} + \frac{\epsilon_1}{h}\left\{0{\cdot}66 + 0{\cdot}58\,\frac{W}{P_2}\right\}\right]\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} (333)$$

or giving to $\frac{W}{P_2}$ and $\frac{F_c}{P''}$ their superior limit $\frac{1}{5}$,

$$f_c = \frac{2W}{a}\left[\frac{1}{2} + 0{\cdot}78\,\frac{\epsilon_1}{h}\right]\left\{1 + 1{\cdot}3\,\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad . \quad . \quad . \quad (333\text{A})$$

**Tertiary Flexure.**—In large columns it is quite common for the flange itself to be constructed of several elements, usually of a number of flange plates, in which case the strength of the flange considered as a column between the panel points, and hence the strength of the column as a whole, depends on the strength of the elementary columns into which the flange may be divided. Such columns usually fail due to the outer flange plate crippling between the rivets (Fig. 36).

In a case such as that shown in Fig. 36, where the width $t$ of the elementary column is small compared with the width of the flange, the load on the elementary column is $= f_c'''a'''$, where $f_c'''$ is the stress in the extreme fibres as given in equation (312), (325), or (333), and $a'''$ is the area of the elementary column. This elementary column may be assumed to be imperfectly direction-fixed at the ends, in which case the maximum stress at the centre will be given by equation (205A), which becomes

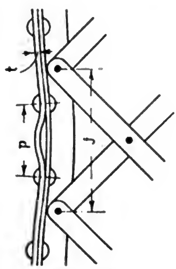




FIG. 36.

$$f_c = \frac{f'''a'''}{a'''}\left[1 + 0{\cdot}76\,\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right] = f_c'''\left[1 + 0{\cdot}76\,\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right] \quad . \quad . \quad . \quad (334)$$

where $\epsilon_1'''$, $v_2'''$, and $\kappa'''$ have reference to the elementary column. The maximum stress at the ends will be given by equation (187) if the worst possible

assumptions be made, namely that the direction-fixing is perfect and $k = 1$. This equation becomes

$$f_c = f_c''' \left[1 + 0{\cdot}78 \frac{\epsilon_1'''v_1'''}{(\kappa''')^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (335)$$

If $v_1''' = v_2'''$, the two equations (334) and (335) will give practically identical results; since the stress given by the latter is admittedly too large, the former will be adopted.

For position-fixed columns, therefore, if the value of $f_c'''$ from equation (312) be inserted in (334), the expression for the maximum stress at the centre of the column becomes

$$f_c = \frac{2W}{a}\left\{\frac{1}{2} + \left(1 + \frac{3W}{2P}\right)\frac{\epsilon_1 + \epsilon_2}{h}\right\}\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}$$
$$\times\left\{1 + 0{\cdot}76\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (336)$$

or, giving to $\frac{W}{P}$ and $\frac{F_c}{P''}$ their superior limit $\frac{1}{5}$,

$$f_c = \frac{2W}{a}\left\{\frac{1}{2} + 1{\cdot}3\frac{\epsilon_1 + \epsilon_2}{h}\right\}\left\{1 + 1{\cdot}3\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}\left\{1 + 0{\cdot}76\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right\} \quad (336\text{A})$$

For position- and direction-fixed columns, if the value of $f_c'''$ from equation (325) be inserted in (334), the expression for the maximum stress at the centre of the column becomes

$$f_c = \frac{2W}{a}\left[\frac{1}{2} + \frac{2k\epsilon_1}{h}\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2 W}\left(1 - \frac{1}{k}\right)\right\}\right]$$
$$\times\left\{1 + \left(1 + \frac{3F_c}{2P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}\left\{1 + 0{\cdot}76\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right\} (337)$$

and from equation (333) the expression for the maximum stress at the ends of the column becomes

$$f_c = \frac{2W}{a}\left[\frac{1}{2} + \frac{\epsilon_1}{h}\left\{0{\cdot}66 + 0{\cdot}58\frac{W}{P_2}\right\}\right]\left\{1 + \left(1 + \frac{3F_c}{P''}\right)\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\}$$
$$\times\left\{1 + 0{\cdot}76\frac{\epsilon_1'''v_2'''}{(\kappa''')^2}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (338)$$

If the flange is composed of flange plates, and the outer one buckles (Fig. 36), the factor $\frac{v_2'''}{(\kappa''')^2}$ is equal to $\frac{6}{t}$, where $t$ is the thickness of the outer plate. The value of $\epsilon'''$ should be taken as equal to that for an unstraightened specimen, say twice the probable value for a straightened specimen. The latter value is shown in Chapter III to be equal to $\frac{L}{750}$, hence $\epsilon_1'''$ may be taken as equal to $\frac{p}{375}$ (Fig. 36). Inserting these values in equation (334),

$$f_c = f_c'''\left[1 + 0{\cdot}76\frac{6}{t}\cdot\frac{p}{375}\right]$$
$$= f_c'''\left[1 + \frac{p}{82{\cdot}2t}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339)$$

or in round figures

$$f_c = f_c''' \left[1 + \frac{p}{80t}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (340)$$

The factor $\left[1 + \frac{p}{80t}\right]$ may in this case be substituted for $\left\{1 + 0.76\frac{\epsilon_1''' \, v_2'''}{(\kappa''')^2}\right\}$ in equations (336), (337), and (338).

For the above equations to hold, the load $f_c'''a'''$ on one of the elementary flange columns must not exceed $\frac{1}{5}$ of Euler's crippling load. That is to say, the maximum value of

$$f_c''' = \frac{\pi^2 E(\kappa''')^2}{5(qp)^2}.$$

Taking the value of $q$ as 0·78

$$\frac{t^2}{p^2} = 3.7\frac{f_c'''}{E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (341)$$

an equation which determines the limiting value of the ratio $\frac{t}{p}$.

In the above equations no attempt has been made to substitute the more exact values for the cosecant and cotangent. It is doubtful if the method is sufficiently accurate to warrant the extra complication. Care should be taken, however, that the appropriate value of $v_2''$ is inserted in the equations, for it is obvious that unless the flange plate lie on the concave side of the elementary flange column, tertiary flexure is impossible.

## *NON-UNIPLANAR BENDING. SOLID COLUMNS*

### CASE VII. Position-fixed Columns

*Both ends fixed in position but free in direction*

#### THE ORDINARY COLUMN

It has been seen that the imperfections in a position-fixed column may be taken into account by assuming the line of resistance to have an initial deflection $\epsilon_1$ and the load an eccentricity $\epsilon_2$. In the case of uniplanar bending it was assumed that all the imperfections produced flexure in the same direction. In general this will not be the case, and the bending will not be uniplanar.

Let Fig. 37 be a plan of the column; BAB, which appears as a point, being the load line. Take origin at A, the centre point of the line BB. Let A$x$, A$y$, A$z$ be the axes of co-ordinates; A$x$ being the load line, A$y$ and A$z$ being drawn parallel to the principal axes of elasticity of the cross section.

Then BU $= \epsilon_2$ is the eccentricity of loading, UU$_1 = \epsilon_1$ is the initial deflection. Let K$_1$, whose co-ordinates are $xy_1z_1$, be any point on the line of resistance in its original position, and suppose that owing to the application of the load W the point K$_1$ moves to K$_0$ and U$_1$ to U$_0$. Let the co-ordinates of K$_0$ be $xyz$, and of U$_0$ O$y_0z_0$. It will be supposed that the line UK$_1$U$_1$U is a smooth curve, and for convenience it will be assumed that its projections on the planes $xy$ and $xz$ are parabolas, the exact shape of the curves, which are supposed to be very flat, not being of great importance.

Then, from equation (82), the equation to the projection of the line $UU_1U$ on to the plane $xy$ is

$$y_1 = \epsilon_2 \cos \phi_2 + \epsilon_1 \cos \phi_1 \left(1 - \frac{4x^2}{L^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (342)$$

from which it follows, exactly as in Case I, Variation 6, equations (84) and (85),

that $$y = \left(\epsilon_2 \cos \phi_2 + \frac{8\epsilon_1}{a_y{}^2L^2} \cos \phi_1\right) \sec \frac{a_yL}{2} \cos a_yx - \frac{8\epsilon_1}{a_y{}^2L^2} \cos \phi_1 \quad . \quad (343)$$

and $$y_0 = \left(\epsilon_2 \cos \phi_2 + \frac{8\epsilon_1}{a_y{}^2L^2} \cos \phi_1\right) \sec \frac{a_yL}{2} - \frac{8\epsilon_1}{a_y{}^2L^2} \cos \phi_1 \quad . \quad . \quad (344)$$

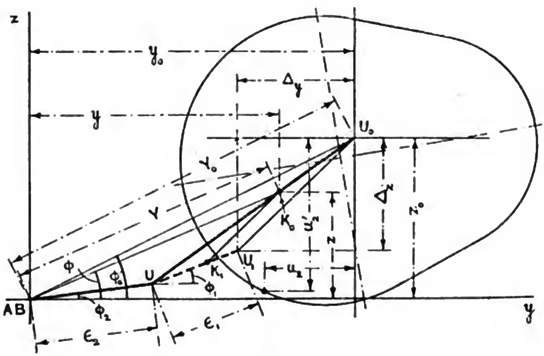



FIG. 37.

where $a_y{}^2 = \frac{W}{S_y(1 - s_a)}$, and $S_y$ is the moment of stiffness about the principal axis of elasticity perpendicular to the plane $xy$.

(It should be remarked that all quantities with a suffix $y$ have reference to bending in the plane $xy$, and those with a suffix $z$ to bending in the plane $xz$.)

Similarly, for the plane $xz$,

$$z_1 = \epsilon_2 \sin \phi_2 + \epsilon_1 \sin \phi_1 \left(1 - \frac{4x^2}{L^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (345)$$

$$z = \left(\epsilon_2 \sin \phi_2 + \frac{8\epsilon_1}{a_z{}^2L^2} \sin \phi_1\right) \sec \frac{a_zL}{2} \cos a_zx - \frac{8\epsilon_1}{a_z{}^2L^2} \sin \phi_1 \quad . \quad (346)$$

and $$z_0 = \left(\epsilon_2 \sin \phi_2 + \frac{8\epsilon_1}{a_z{}^2L^2} \sin \phi_1\right) \sec \frac{a_zL}{2} - \frac{8\epsilon_1}{a_z{}^2L^2} \sin \phi_1 \quad . \quad . \quad (347)$$

From these equations the value and direction of the total deflection Y can be obtained.

$$Y = \sqrt{y^2 + z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (348)$$

$$\tan \phi = \frac{z}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (349)$$

At the centre of the column $Y = Y_0$, $\phi = \phi_0$, $y = y_0$, and $z = z_0$. Hence

$$Y_0 = \sqrt{y_0^2 + z_0^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (350)$$

$$\tan \phi_0 = \frac{z_0}{y_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (351)$$

Knowing $Y_0$ and $\phi_0$, the maximum stresses in the material may be obtained. If the modulus of elasticity be constant everywhere, use may be made of the properties of the core. The position of the neutral axis may be found, and hence the point of maximum stress. In this case

$$f_c = f_a\left[1 + \frac{Y_0}{\omega_2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (352)$$

(see Tetmajer, 1896).

If the modulus of elasticity vary, the simple core theory cannot be used, and the solution in the general case is more difficult. In many practical cases, however, the probable points where the stress will be a maximum may be found by inspection, the stresses due to bending parallel to the planes $xy$ and $xz$ may be found, and their combination with the direct stress determines the maximum stress in the material. Thus in a section such as is shown in Fig. 38 the maximum compressive stress at the centre of the column will occur at the point $H_2$.

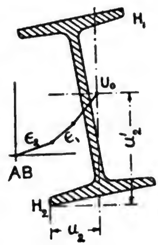



FIG. 38.

If bending took place solely in the plane $xy$, the stress at $H_2$ would be given by equation (91), which, for the case under consideration, becomes

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa_y^2}\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2 \cos\phi_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P_y}} + \epsilon_1 \cos\phi_1 \frac{8P_y}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P_y}} - 1\right)\right\}\right]$$

where $\kappa_y$ and $P_y$ are the values of $\kappa$ and P having reference to bending in the plane $xy$.

If, on the other hand, bending took place solely in the plane $xz$, the expression for the stress at $H_2$ would be

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa_z^2}\left(v_2' + e\frac{a_1'\bar{v}_1'}{a}\right)\left\{\epsilon_2 \sin\phi_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P_z}} + \epsilon_1 \sin\phi_1 \frac{8P_z}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P_z}} - 1\right)\right\}\right]$$

where $\kappa_z$, $P_z$, $v_2'$, $a_1'$, and $\bar{v}_1'$ have reference to bending in the plane $xz$, the appropriate value of $e$ being determined by the given variation in the modulus of elasticity.

If, however, as will in general be the case, bending take place in both directions at the same time, the expression for the maximum stress at $H_2$ is

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \frac{1}{\kappa_y^2}\left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\left\{\epsilon_2 \cos\phi_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P_y}} + \epsilon_1\cos\phi_1\frac{8P_y}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P_y}} - 1\right)\right\} + \frac{1}{\kappa_z^2}\left(v_2' + e\frac{a_1\bar{v}_1'}{a}\right)\left\{\epsilon_2\sin\phi_2\sec\frac{\pi}{2}\sqrt{\frac{W}{P_z}} + \epsilon_1\sin\phi_1\frac{8P_z}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P_z}} - 1\right)\right\}\right] \quad . \quad (353)$$

This equation may be simplified in the manner adopted to simplify equation (91); the factors containing $e$ may be neglected, and the secants replaced by suitable straight lines (see p. 43). The expression then becomes

$$f_c = \frac{W}{a}\left[1 + \frac{v_2}{\kappa_y^2}\left\{(\epsilon_1\cos\phi_1 + \epsilon_2\cos\phi_2)\left(1 + \frac{3W}{2P_y}\right)\right\} + \frac{v_2'}{\kappa_z^2}\left\{(\epsilon_1\sin\phi_1 + \epsilon_2\sin\phi_2)\left(1 + \frac{3W}{2P_z}\right)\right\}\right] \quad . \quad . \quad . \quad (354)$$

Giving to $\frac{W}{P_y}$ and $\frac{W}{P_z}$ their superior limit $\frac{1}{5}$, equation (354) becomes

$$f_c = \frac{W}{a}\left[1 + 1{\cdot}3\left\{\frac{v_2}{\kappa_y^2}(\epsilon_1\cos\phi_1 + \epsilon_2\cos\phi_2) + \frac{v_2'}{\kappa_z^2}(\epsilon_1\sin\phi_1 + \epsilon_2\sin\phi_2)\right\}\right] \quad (355)$$

a first approximation to the stress at the point $H_2$.

Equations (354) and (355) apply only when the ratio of $\frac{W}{P}$ in both directions is less than $\frac{1}{5}$, that is to say, under working conditions.

*Lattice-braced Columns.*—In a lattice-braced column, where secondary and tertiary flexure occurs, it is necessary to find the stress $f_c$ in the plane $xy$ from equations (312) or (336) as the case may be, and to add the stress due to *bending* in the plane $xz$ from equation (93A), taking care to neglect the factor unity representing the direct stress in the latter equation.

## CASE VIII. Position- and Direction-fixed Columns

*Both ends fixed in position and direction*

### THE ORDINARY COLUMN

In addition to the imperfections usual in position-fixed columns, there is, in the case of position- and direction-fixed columns, the effect of imperfect direction-fixing to be allowed for. As in Case II, Variation 7, the whole of the imperfections can be taken into account by assuming the column to be initially curved and eccentrically loaded, and that, due to the imperfect direction-fixing, a given increase in the slope at the ends of the column takes place.

In general the various imperfections will tend to produce flexure in different planes, and the bending will not be uniplanar.

As in Case VII, Fig. 37, let $\epsilon_1$ be the original deflection, and $\epsilon_2$ the eccentricity of loading. Suppose also that the line of resistance in its original position is a smooth curve, and that its projections on to the planes *xy* and *xz* are parabolas.

Then, as in equation (342), the equation to the projection of the line $UU_1U$ on to the plane *xy* is

$$y_1 = \epsilon_2 \cos\phi_2 + \epsilon_1 \cos\phi_1\left(1 - \frac{4x^2}{L^2}\right).$$

Hence $\dfrac{dy_1}{dx} = -\dfrac{4\epsilon_1}{L}\cos\phi_1$ when $x = \dfrac{L}{2}$.

This is the original slope of the projection of the line of resistance at its ends. Suppose that this increases to $-\dfrac{4k\epsilon_1}{L}\cos\phi_1$ owing to the application of the load and the imperfection in the direction-fixing. The coefficient $k$ may vary with the load, its value being somewhat greater than unity. Then it follows, exactly as in Case II, Variation 7, equations (166), (167), and (169), that

$$M_s = -\frac{8W\epsilon_1\cos\phi_1}{a_y^2L^2}\left\{1 - k\frac{a_yL}{2}\cot\frac{a_yL}{2}\right\} - W\epsilon_2\cos\phi_2 \quad . \quad (356)$$

$$y_0 = \frac{4k\epsilon_1\cos\phi_1}{a_yL}\left(\frac{1-\cos\frac{a_yL}{2}}{\sin\frac{a_yL}{2}}\right) + \epsilon_2\cos\phi_2 \quad . \quad . \quad . \quad . \quad . \quad (357)$$

$$y = \frac{4k\epsilon_1\cos\phi_1}{a_yL}\left(\frac{\cos a_yx - \cos\frac{a_yL}{2}}{\sin\frac{a_yL}{2}}\right) + \epsilon_2\cos\phi_2 \quad . \quad . \quad . \quad (358)$$

where $a_y^2 = \dfrac{W}{S_y(1-s_a)}$, and $S_y$ is the moment of stiffness about the principal axis of elasticity perpendicular to the plane *xy*.

Similarly for the plane *xz*

$$M_a' = -\frac{8W\epsilon_1\sin\phi_1}{a_z^2L^2}\left\{1 - k'\frac{a_zL}{2}\cot\frac{a_zL}{2}\right\} - W\epsilon_2\sin\phi_2 \quad . \quad . \quad (359)$$

$$z_0 = \frac{4k'\epsilon_1\sin\phi_1}{a_zL}\left(\frac{1-\cos\frac{a_zL}{2}}{\sin\frac{a_zL}{2}}\right) + \epsilon_2\sin\phi_2 \quad . \quad . \quad . \quad . \quad . \quad (360)$$

$$z = \frac{4k'\epsilon_1\sin\phi_1}{a_zL}\left(\frac{\cos a_zx - \cos\frac{a_zL}{2}}{\sin\frac{a_zL}{2}}\right) + \epsilon_2\sin\phi_2 \quad . \quad . \quad . \quad . \quad (361)$$

where $M_a'$ is the value of the direction-fixing moment, and $k'$ the value of the coefficient $k$, having reference to bending in the plane $xz$.

From these equations the value and direction of the total deflection Y can be obtained.

$$Y = \sqrt{y^2 + z^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (362)$$

$$\tan \phi = \frac{z}{y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (363)$$

At the centre of the column $Y = Y_0$, $\phi = \phi_0$, $y = y_0$, and $z = z_0$. Hence

$$Y_0 = \sqrt{y_0^2 + z_0^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (364)$$

$$\tan \phi_0 = \frac{z_0}{y_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (365)$$

The total fixing moment at the ends of the column is equal to $\sqrt{M_a^2 + (M_a')^2}$

The stresses in the material anywhere are best determined by finding the stresses due to bending in the two planes $xy$ and $xz$ separately, and then combining them with the direct stress.

In a section such as that shown in Fig. 38, the maximum compressive stress at the centre of the column will occur at the point $H_2$. If bending took place solely in the plane $xy$, the stress at $H_2$ would be given by equation (175), which, for the case under consideration, becomes

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1 \cos\phi_1}{\pi^2\kappa_y^2}\frac{P_y''}{W}\right.$$
$$\left.\times\left\{k\,\pi\sqrt{\frac{W}{P_y''}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_y''}} - 1\right\}\right] \qquad (366)$$

where $\kappa_y$ and $P_y''$ are the values of $\kappa$ and $P_2$ having reference to bending in the plane $xy$.

If, on the other hand, bending took place solely in the plane $xz$, the expression for the stress at $H_2$ would be

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2' + e\frac{a_1'\bar{v}_1'}{a}\right)\frac{2\epsilon_1 \sin\phi_1}{\pi^2\kappa_z^2}\frac{P_z''}{W}\right.$$
$$\left.\times\left\{k'\,\pi\sqrt{\frac{W}{P_z''}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_z''}} - 1\right\}\right] \qquad (367)$$

where $\kappa_z$, $P_z''$, $k'$, $v_2'$, $a_1'$, and $\bar{v}_1'$ have reference to bending in the plane $xz$, the appropriate value of $e$ being determined by the given variation in the modulus of elasticity.

If, however, as will in general be the case, bending take place in both directions at the same time, the expression for the maximum stress at $H_2$ is

$$f_c = \frac{W}{a}\left(1 - \frac{e}{2}\right)\left[1 + \left(v_2 + e\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1 \cos\phi_1}{\pi^2\kappa_y^2}\frac{P_y''}{W}\right.$$
$$\times\left\{k\,\pi\sqrt{\frac{W}{P_y''}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_y''}} - 1\right\} + \left(v_2' + e\frac{a_1'\bar{v}_1'}{a}\right)\frac{2\epsilon_1 \sin\phi_1}{\pi^2\kappa_z^2}\frac{P_z''}{W}$$
$$\left.\times\left\{k'\,\pi\sqrt{\frac{W}{P_z''}}\operatorname{cosec}\pi\sqrt{\frac{W}{P_z''}} - 1\right\}\right] \quad . \quad . \quad . \quad . \quad . \qquad (368)$$

This equation may be simplified in the manner adopted to simplify equation (175); the factors containing $e$ may be neglected, and the cosecants replaced by suitable straight lines [equation (178)]. The expression then becomes [see equation (181)]

$$f_c = \frac{W}{a}\left[1 + \frac{2k\epsilon_1 \cos\phi_1 v_2}{\kappa_y^2}\left\{0\cdot17 + 0\cdot26\frac{W}{P_y''} + \frac{1}{\pi^2}\frac{P_y''}{W}\left(1 - \frac{1}{k}\right)\right\}\right.$$
$$\left. + \frac{2k'\epsilon_1 \sin\phi_1 v_2'}{\kappa_z^2}\left\{0\cdot17 + 0\cdot26\frac{W}{P_z''} + \frac{1}{\pi^2}\frac{P_z''}{W}\left(1 - \frac{1}{k'}\right)\right\}\right] \quad . \quad . (369)$$

Giving to $\frac{W}{P_y''}$ and $\frac{W}{P_z''}$ their probable superior limit $\frac{1}{5}$, equation (369) becomes

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 \cos\phi_1 v_2}{\kappa_y^2}\left\{1\cdot44k - 1\right\} + \frac{\epsilon_1 \sin\phi_1 v_2'}{\kappa_z^2}\left\{1\cdot44k' - 1\right\}\right] \quad (370)$$

If the direction fixing be perfect, $k$ and $k' = 1$, when

$$f_c = \frac{W}{a}\left[1 + 0\cdot44\left\{\frac{\epsilon_1 \cos\phi_1 v_2}{\kappa_y^2} + \frac{\epsilon_1 \sin\phi_1 v_2'}{\kappa_z^2}\right\}\right]. \quad . \quad . \quad (371)$$

Equations (369) to (371) apply only when the ratio $\frac{W}{P_2}$ in both directions is less than $\frac{1}{5}$, that is to say, under working conditions. If $k$ and $k'$ be large, equation (369) should be used and W limited to a value less than $\frac{\pi^2 EI_y}{5(qL)^2}$ or $\frac{\pi^2 EI_z}{5(qL)^2}$.

In a section such as that shown in Fig. 38, the maximum compressive stress at the ends of the column will occur at the point $H_1$. If bending took place solely in the plane $xy$, the stress at $H_1$ would be given by equation (177), which, for the case under consideration, becomes

$$f_c = \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[1 + \left(v_1 - e_6\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1 \cos\phi_1}{\pi^2\kappa_y^2}\frac{P_y''}{W}\right.$$
$$\left.\times\left\{1 - k\pi\sqrt{\frac{W}{P_y''}}\cot\pi\sqrt{\frac{W}{P_y''}}\right\}\right].$$

If, on the other hand, bending took place solely in the plane $xz$, the expression for the stress at $H_1$ would be

$$f_c = \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[1 + \left(v_1' - e_6\frac{a_1'\bar{v}_1'}{a}\right)\frac{2\epsilon_1 \sin\phi_1}{\pi^2\kappa_z^2}\frac{P_z''}{W}\right.$$
$$\left.\times\left\{1 - k'\pi\sqrt{\frac{W}{P_z''}}\cot\pi\sqrt{\frac{W}{P_z''}}\right\}\right].$$

The appropriate value of $e_6$ must be determined from the given variation in the modulus of elasticity. The significance of the other symbols is defined above.

If, as in general will be the case, bending take place in both directions at the same time, the expression for the maximum stress at $H_1$ is

$$f_c = \frac{W}{a}\left(1 + \frac{e_6}{2}\right)\left[1 + \left(v_1 - e_6\frac{a_1\bar{v}_1}{a}\right)\frac{2\epsilon_1 \cos\phi_1}{\pi^2\kappa_y^2}\frac{P_y''}{W}\right.$$
$$\times\left\{1 - k\pi\sqrt{\frac{W}{P_y''}}\cot\pi\sqrt{\frac{W}{P_y''}}\right\} + \left(v_1' - e_6\frac{a_1'\bar{v}_1'}{a}\right)\frac{2\epsilon_1 \sin\phi_1}{\pi^2\kappa_z^2}\frac{P_z''}{W}$$
$$\left.\times\left\{1 - k'\pi\sqrt{\frac{W}{P_z''}}\cot\pi\sqrt{\frac{W}{P_z''}}\right\}\right] \quad . \quad . \quad . \quad . \quad . \quad (372)$$

This equation may be simplified in the manner adopted to simplify equation (177); the factors containing $e_6$ can be neglected, and the cotangents replaced by suitable straight lines [equation (179)]. The expression then becomes [see equation (184)]

$$f_c = \frac{W}{a}\left[1 + \frac{2k\epsilon_1 \cos\phi_1 v_1}{\kappa_y^2}\left\{0{\cdot}33 + 0{\cdot}29\frac{W}{P_y''} - \frac{1}{\pi^2}\frac{P_y''}{W}\left(1 - \frac{1}{k}\right)\right\}\right.$$
$$\left. + \frac{2k'\epsilon_1 \sin\phi_1 v_1'}{\kappa_z^2}\left\{0{\cdot}33 + 0{\cdot}29\frac{W}{P_z''} - \frac{1}{\pi^2}\frac{P_z''}{W}\left(1 - \frac{1}{k'}\right)\right\}\right] \quad . \quad (373)$$

Giving to $\frac{W}{P_y''}$ and $\frac{W}{P_z''}$ their probable superior limits $\frac{1}{5}$, equation (373) becomes

$$f_c = \frac{W}{a}\left[1 + \frac{\epsilon_1 \cos\phi_1 v_1}{\kappa_y^2}\left\{1 - 0{\cdot}22\,k\right\} + \frac{\epsilon_1 \sin\phi_1 v_1'}{\kappa_z^2}\left\{1 - 0{\cdot}22\,k'\right\}\right] (374)$$

Putting, as the worst possible assumption, $k$ and $k'$ equal to unity,

$$f_c = \frac{W}{a}\left[1 + 0{\cdot}78\left\{\frac{\epsilon_1 \cos\phi_1 v_1}{\kappa_y^2} + \frac{\epsilon_1 \sin\phi_1 v_1'}{\kappa_z^2}\right\}\right] . \quad . \quad . \quad (375)$$

Equations (373) to (375) apply only when the ratio $\frac{W}{P_2}$ in both directions is less than $\frac{1}{5}$, that is to say, under working conditions.

In a lattice-braced column, where secondary and tertiary flexure occurs, it is necessary to find the stress $f_c$ in the plane $xy$ from equations (325), (333), (337), or (338), as the case may be, and to add the stress due to *bending* in the plane $xz$ from equations (181) or (186) corresponding, taking care in the latter equations to neglect the factor unity representing the direct stress.

## COLUMNS WITH LATERAL LOADS

The problem of columns with lateral loads is chiefly of importance in horizontal members, where the proper weight of the member tends to increase its deflection. In aeroplane struts the wind pressure forms a lateral load on the member. Another instance in which the lateral load may be important is that of a member position- and direction-fixed at its lower end, free at its upper end, and loaded there with a force not parallel to its axis. In this case there is a transverse or lateral component.

This latter problem was treated by Lagrange (1771), Navier (1833), and Bressé (1859); Ritter (1874), Saalschütz (1880), and others have attacked the same question. Saalschütz has given a strict investigation regarding the shape of the elastic line under such loading. The case of a column with a uniform lateral load has more recently received a certain amount of attention, in England, America, and Germany.

The following is a list of the more important work on this branch of the subject.

LAGRANGE.—*Sur la force des ressorts pliés.* Mémoires de l'Académie de Berlin. 1771.
NAVIER.—*Résumé des leçons données à l'école des ponts et chaussées.* Part i, second edition, 1833.
WIESBACH.—*Lehrbuch der Ing.- u. Mach.-Mechanik.* 1855.
SCHEFFLER.—*Theorie der Festigkeit gegen das Zerknicken.* 1858.
BRESSE.—*Cours de mécanique appliquée.* Première partie, 1859.
WINKLER.—*Die Lehre von der Elasticität und Festigkeit.* 1867, Part i.
RITTER.—*Lehrbuch der Technischen Mechanik.* Third edition, 1874.
SAALSCHÜTZ.—*Der Belastete Stab.* 1880.
PERRY.—*Struts with Lateral Loads.* Proc. Physical Society. Vol. xi, Dec. 1891, and Philosophical Magazine, March 1892.
FRANCKE.—*Die Zerknickungsfestigkeit.* Zeits. des Arch.- u. Ing.-Vereines zu Hannover. 1895. Heft 8, p. 622.
HEAD.—*The Problem of Struts with Lateral Loads.* The Engineer. London, Sept. 22, 1899.
KRIEMLER.—*Labile und Stabile Gleichgewichtsfiguren.* 1902.
MERRIMAN.—*Mechanics of Materials.* Tenth edition, 1905.
VIANELLO.—*Der Eisenbau.* 1905.
MORLEY.—*Laterally Loaded Struts and Ties.* Philosophical Magazine. June, 1908.
LILLY.—*Eccentrically Loaded Columns.* Proc. Inst. C.E. Vol. clxxxi, 1910.
WITTENBAUER.—*Aufgaben der Technischen Mechanik.* 1910.
KAYSER.—*Auf Biegung beanspruchte Druckstäbe.* Zentralblatt der Bauverwaltung. June 1910, p. 304.
HUTT.—*The Theoretical Principles of Strut Design.* Engineering. London, Aug. 2, 1912.
BARLING and WEBB.—*Design of Aeroplane Struts.* Aero. Journal. London, Oct. 1918. Also Rpts. and Mem. of Advis. Com. for Aeronautics. Nos. 343, 363.
ARNSTEIN.—*Beanspruchung axial gedrückter durch einzellasten gebogener Stäbe.* Eisenbau. Leipzig, 1919, p. 151.

The problem of the laterally loaded continuous column has arisen in connexion with aeroplane wing spars. A general solution was given by H. BOOTH and H. BOLAS: *Some Contributions to the Theory of Engineering Structures with Special Reference to the Problem of the Aeroplane, Admir. Con. Mem. (Air Dept.)*, April 1915. Values for the trigonometrical functions were calculated by H. BERRY (Berry Functions), *Admir. Con. Mem. (Air Dept.)*, July, 1916, who adapted the work for practical use. See *The Calculation of Stresses in Aeroplane Wing Spars. Trans. Roy. Aero. Soc.*, No. 1. London, 1919. Similar results appear to have been obtained in Germany by H. MÜLLER-BRESLAU. See his *Graphische Statik*, and also *Tech.-Berichte h. v. d. Flugzeugmeisterei*, Aug. 1918. In this connexion the following should also be consulted.

COWLEY and LEVY.—*Critical Loading of Struts and Structures.* Proc. Roy. Soc. London, Series A, vol. xciv, p. 405.
WEBB and THORN.—*Wing Spar Stresses.* Aeronautics. London, Jan. 1, 1919, p. 8.
L. N. G. FILON.—*Investigation of Stresses in Aeroplane Wing Framework.* Brit. Assoc. Rept., 1919. London, 1920, p. 468.

*Column with a Uniform Lateral Load.*—The following is the usual solution to the problem of a uniform, originally straight, concentrically loaded, homogeneous solid column, position-fixed at both ends and loaded with a uniform lateral load of $w$ per unit run. The bending is supposed to be uniplanar.

Let $UU_0U$, Fig. 39, be the shape of the line of resistance of the bent column (its central axis). Take origin at A in the line of action of the load, and let $AU = \frac{L}{2}$. Consider any point $K_0$ in the line of resistance. Let the co-ordinates of $K_0$ be $x$ and $y$. Then the bending moment at $K_0$ is

$$M = Wy + \frac{wL^2}{8} - \frac{wx^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (376)$$

Hence, from equation (15),

$$\frac{d^2y}{dx^2} + \frac{1}{I\left(E - \frac{W}{a}\right)}\left\{Wy + \frac{wL^2}{8} - \frac{wx^2}{2}\right\} = 0 \quad . \quad . \quad . \quad . \quad (377)$$

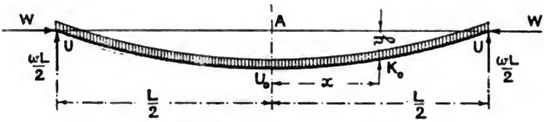



FIG. 39.

Let
$$\frac{W}{I\left(E - \frac{W}{a}\right)} = a^2.$$

Then
$$\frac{d^2y}{dx^2} + a^2y + a^2\frac{wL^2}{8W} - a^2\frac{wx^2}{2W} = 0,$$

to which the solution is

$$y = m \sin ax + n \cos ax - \frac{wL^2}{8W} + \frac{wx^2}{2W} - \frac{w}{a^2W}.$$

But when $x = 0$, $\frac{dy}{dx} = 0$, and hence $m = 0$. Also when $x = \frac{L}{2}$, $y = 0$, therefore

$$0 = n \cos\frac{aL}{2} - \frac{wL^2}{8W} + \frac{wL^2}{8W} - \frac{w}{a^2W}$$

or
$$n = \frac{w}{a^2W} \sec \frac{aL}{2}.$$

Hence
$$y = \frac{w}{W}\left\{\frac{1}{a^2}\left(\sec\frac{aL}{2}\cos ax - 1\right) + \left(\frac{x^2}{2} - \frac{L^2}{8}\right)\right\} \quad . \quad . \quad . \quad (378)$$

The maximum value of $y$ occurs when $x = 0$, and is

$$y_0 = \frac{w}{W}\left\{\frac{1}{a^2}\left(\sec\frac{aL}{2} - 1\right) - \frac{L^2}{8}\right\} \quad . \quad . \quad . \quad . \quad . \quad (379)$$

[Compare equation (30) for parabolic initial curvature.]

The bending moment anywhere, from equation (376), is

$$M = Wy + \frac{wL^2}{8} - \frac{wx^2}{2}$$

$$= w\left\{\frac{1}{a^2}\left(\sec\frac{aL}{2}\cos ax - 1\right) + \left(\frac{x^2}{2} - \frac{L^2}{8}\right)\right\} + \frac{wL^2}{8} - \frac{wx^2}{2}$$

$$= \frac{w}{a^2}\left(\sec\frac{aL}{2}\cos ax - 1\right)$$

$$= \frac{wI}{W}\left(E - \frac{W}{a}\right)\left(\sec\frac{L}{2}\cos ax - 1\right) \quad . \quad . \quad . \quad . \quad (380)$$

This becomes a maximum when $x = O$.

$$M_{max} = \frac{w}{a^2}\left(\sec\frac{aL}{2} - 1\right)$$

$$= \frac{wI}{W}\left(E - \frac{W}{a}\right)\left(\sec\frac{aL}{2} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad (381)$$

The maximum compressive stress $f_c$ will occur at the centre

$$f_c = \frac{W}{a} + \frac{Mv_2}{I}$$

$$= \frac{W}{a} + \frac{wv_2}{Ia^2}\left(\sec\frac{aL}{2} - 1\right)$$

$$= \frac{W}{a} + \frac{wv_2}{W}\left(E - \frac{W}{a}\right)\left(\sec\frac{aL}{2} - 1\right) \quad . \quad . \quad . \quad . \quad (382)$$

If $\frac{W}{a}$ be neglected in comparison with E, $a^2 = \frac{W}{EI}$ and $\frac{aL}{2} = \frac{\pi}{2}\sqrt{\frac{W}{P}}$. The value of $f_c$ then becomes

$$f_c = \frac{W}{a} + \frac{Ewv_2}{W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right) . \quad . \quad . \quad . \quad . \quad (383)$$

Expanding the secant,

$$\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right) = \left(\sec\sqrt{\frac{WL^2}{4EI}} - 1\right)$$

$$= \frac{1}{2}\left(\frac{WL^2}{4EI}\right) + \frac{5}{24}\left(\frac{WL^2}{4EI}\right)^2 + \frac{1}{12}\left(\frac{WL^2}{4EI}\right)^3 + \ . \ . \ .$$

Hence the expression for the stress may be written

$$f_c = \frac{W}{a} + \frac{wL^2}{8Z} + \frac{5}{384}\cdot\frac{wL^4}{EI}\cdot\frac{W}{Z} + \ . \ . \ . \qquad (384)$$

It will be observed that the first of these terms in order is the stress due to direct compression. The second is the stress due to the distributed load separately considered. The third is the bending stress due to the longitudinal load W acting on a column bent to the deflection caused by the distributed load $w$. The remaining terms of the series represent the increment in deflection and stress due to W and $w$ acting together.

Head * replaces the above series by a geometrical progression. He writes the expression for $y_0$

$$y_0 = \frac{5}{24} \cdot \frac{wL^4}{16EI} \left\{ 1 + \frac{24}{5} \cdot \frac{1}{12} \left( \frac{WL^2}{4EI} \right) + \frac{24}{5} \cdot \frac{19}{576} \left( \frac{WL^2}{4EI} \right)^2 + \ . \ . \ . \right\}$$

This is convergent if $\frac{2}{5} \cdot \frac{WL^2}{4EI}$ be less than unity. That is, if

$$\frac{WL^2}{4EI} < \frac{5}{2} < \frac{\pi^2}{4}, \text{ or } W < \frac{\pi^2 EI}{L^2}.$$

Hence it follows that the deflection will be definite unless $W > P$, and therefore the lateral load does not detract from the stability of the column, but merely increases the compressive stress.

The above expression for $y_0$ may therefore be replaced by

$$y_0 = \frac{5}{24} \cdot \frac{wL^4}{16EI} \left\{ 1 + \frac{WL^2}{10EI} + \left( \frac{WL^2}{10EI} \right)^2 + \ . \ . \ . \right\}$$

a geometrical series, which summed to infinity gives

$$y_0 = \frac{5}{24} \cdot \frac{wL^4}{16EI} \left\{ \frac{1}{1 - \frac{WL^2}{10EI}} \right\} \qquad (385)$$

which might be written

$$y_0 = \frac{5}{24} \cdot \frac{wL^4}{16EI} \cdot \frac{P}{P - W} \qquad (386)$$

Here the deflection due to the transverse load has been increased in the ratio $\frac{P}{P - W}$ to allow for the effect of the longitudinal load W. The maximum value of the bending moment is then

$$M = \frac{wL^2}{8} + \frac{5}{24} \cdot \frac{wL^4}{16EI} \cdot \frac{PW}{P - W} \qquad (387)$$

Another modification of the above analysis, due to Perry,† may also be noted. Perry replaces the differential equation (377) by

$$\frac{d^2y}{dx^2} + \frac{W}{EI} y + \frac{wL^2}{8EI} \cos \frac{\pi}{L} x = 0 \qquad (388)$$

a not very different function. The solution to this is

$$y = \frac{wL^2}{8(P - W)} \cos \frac{\pi x}{L} \qquad (389)$$

The maximum bending moment occurs at the centre, and is

$$M = Wy_0 + \frac{wL^2}{8}$$

$$= \frac{wL^2}{8} \cdot \frac{P}{P - W} \qquad (390)$$

* *The Engineer*, London, Sept. 22, 1899. † *Philosophical Magazine*, March, 1892.

Hence the maximum compressive stress is

$$f_c = f_a + \frac{wL^2}{8Z} \cdot \frac{P}{P - W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (391)$$

which may be written

$$\left(1 - \frac{f_a}{f_c}\right)\left(1 - \frac{f_a}{f_p}\right) = \frac{wL^2}{8Zf_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (392)$$

Thus according to Head's approximation the central deflection due to the lateral load is increased in the ratio $\frac{P}{P - W}$, while according to Perry's the maximum bending moment at the centre due to the lateral load is increased in the same ratio.

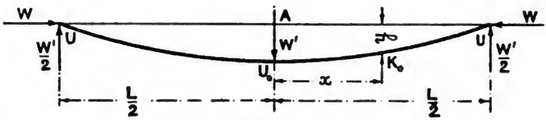



FIG. 40.

*Column with a Central Lateral Load.*—If the lateral load be concentrated at the centre instead of being uniformly distributed (Fig. 40), the differential equation (377) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{I\left(E - \frac{W}{a}\right)}\left\{Wy + \frac{W'}{2}\left(\frac{L}{2} - x\right)\right\} = 0 \quad . \quad . \quad . \quad (393)$$

where $W'$ is the lateral load. Calling $\frac{W}{I\left(E - \frac{W}{a}\right)} = \alpha^2$ as before,

$$\frac{d^2y}{dx^2} + \alpha^2 y + \alpha^2 \cdot \frac{W'L}{4W} - \alpha^2 \cdot \frac{W'x}{2W} = 0,$$

to which the solution is

$$y = m \sin \alpha x + n \cos \alpha x - \frac{W'L}{4W} + \frac{W'x}{2W}.$$

Now when $x = 0$, $\frac{dy}{dx} = 0$, and $m = -\frac{W'}{2\alpha W}$. Further, when $x = \frac{L}{2}$, $y = 0$, and $n = \frac{W'}{2\alpha W} \tan \frac{\alpha L}{2}$. Hence,

$$y = \frac{W'}{2\alpha W} \tan \frac{\alpha L}{2} \cos \alpha x - \frac{W'}{2\alpha W} \sin \alpha x - \frac{W'L}{4W} + \frac{W'x}{2W} \quad . \quad . \quad (394)$$

The maximum value of $y$ occurs at the centre where $x = \frac{L}{2}$, and is

$$y_0 = \frac{W'}{2\alpha W} \tan \frac{\alpha L}{2} - \frac{W'L}{4W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (395)$$

The bending moment anywhere is

$$M = Wy + \frac{W'}{2}\left(\frac{L}{2} - x\right)$$

$$= \frac{W'}{2\alpha}\tan\frac{\alpha L}{2}\cos\alpha x - \frac{W'}{2\alpha}\sin\alpha x \quad . \quad . \quad . \quad . \quad . \quad . \quad (396)$$

This becomes a maximum when $x = O$,

$$M_{max} = \frac{W'}{2\alpha}\tan\frac{\alpha L}{2}$$

$$= \frac{W'}{2}\left\{\frac{I\left(E - \frac{W}{a}\right)}{W}\right\}^{\frac{1}{2}}\tan\frac{L}{2}\left\{\frac{W}{I\left(E - \frac{W}{a}\right)}\right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad (397)$$

The maximum compressive stress $f_c$ will occur at the centre

$$f_c = \frac{W}{a} + \frac{Mv_2}{I}$$

$$= \frac{W}{a} + \frac{W'v_2}{2I\alpha}\tan\frac{\alpha L}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (398)$$

$$= \frac{W}{a} + \frac{W'L}{4}\cdot\frac{v_2}{I}\cdot\frac{2}{\alpha L}\tan\frac{\alpha L}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (399)$$

If $\frac{W}{a}$ be neglected in comparison with E, $\frac{\alpha L}{2} = \frac{\pi}{2}\sqrt{\frac{W}{P}}$ approximately,

and $$f_c = \frac{W}{a} + \frac{W'L}{4}\cdot\frac{v_2}{I}\,\frac{\tan\frac{\pi}{2}\sqrt{\frac{W}{P}}}{\frac{\pi}{2}\sqrt{\frac{W}{P}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (400)$$

The first term is the direct stress, the second is the bending stress due to W′ increased in the ratio

$$\frac{\tan\frac{\pi}{2}\sqrt{\frac{W}{P}}}{\frac{\pi}{2}\sqrt{\frac{W}{P}}}.$$

The value of $\frac{W}{P}$ under working conditions is not likely to exceed $\frac{1}{4}$, hence the probable maximum value of the ratio in question is $\frac{4}{\pi}$.

# PART III

## *SYNTHETICAL*

### CHAPTER III

### THE EULERIAN THEORY

In his celebrated contribution to the theory of columns, Euler (1757) proved that, in the case of a column position-fixed at both ends and originally straight, if the load be directed along the axis of the column, then in order to produce an infinitely small curvature of the column the value of the load must reach the limit $\frac{\pi^2 S}{L^2}$, and if the load be less than this the column will suffer no deflection.

He further showed that if the load exceed this limit, the deflection will be real and increase as the load increases.

Euler's conclusions were confirmed and extended by Lagrange (1770), who gave a more rigid analysis.

His result was so singular that it became the subject of great controversy, which has continued to the present day. Euler himself speaks of it as not a little paradoxical and an apparent interruption of the principle of continuity. He is careful to point out, however, that the paradox is completely cleared up if the difference in length between the chord and the arc be taken into account. If further proof were necessary, Lagrange may be said to have demonstrated even more conclusively that the principle of continuity is completely satisfied.

Nevertheless experiment showed that columns not only bent but actually failed under loads much less than Euler's limiting load. Not only so, but according to Coulomb (1776) the strength of columns is directly proportional to their area and independent of their length, a view endorsed by Rennie (1818); and it was generally concluded that the theory was defective, if not entirely incorrect.

It was evident, in fact, that Euler had neglected the direct compressive stress caused by the load, and even placed the neutral axis on the concave side of the column (Robison, 1822). It was hastily assumed, therefore, that these defects in the theory caused the want of agreement between the formula and experimental results.

The more discerning, however, realized that the differences arose partly, at least, from want of agreement between the assumed and the actual conditions. Young (1807) attributes the irregularities observed in experiments to accidental eccentricity of loading, initial curvature, or inequalities in the material, such that the specimens formed bent rather than straight columns. He points out that under the conditions assumed by Euler there is no reason why the column

should bend, even if the load does exceed the critical value. He further determined a limit below which the column would fail by direct crushing rather than by bending, thus disposing of Coulomb's difficulty. Navier (1833) gave a similar limitation.

The credit for first giving the analysis for eccentrically loaded and initially curved columns is due to Young, and had he been gifted with the power of lucid expression, succeeding generations might have been saved much mathematical disputation.

Meanwhile, attempts were made to correct the Eulerian theory and to take into account the direct stress (Tredgold, 1822; Navier, 1833). The first complete solution appears to have been due to v. Heim (1838). As a by-product of his very general analysis, he finds that if the column is to bend

$$W > \frac{EI}{\frac{L_1^2}{r^2\pi^2} - \frac{I}{a}} = P' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (401)$$

This equation was obtained by the strictest analysis, and includes the effect of the direct compressive stress. The resulting value for the crippling load differs from the Eulerian value by a trifling amount only.

It remained for Kriemler (1902) to include the effect of the shearing force as well as that of the direct compressive stress. By the use of elliptic functions he showed that the limiting value of the load is

$$W \geqq \frac{r^2\pi^2 EI}{L_1^2} \cdot \frac{1}{1 + \frac{W}{a}\left(\frac{\zeta}{G} - \frac{1}{E}\right)} = P'' \quad . \quad . \quad . \quad (402)$$

Now this value, like $P'$, differs very slightly from $P$; in fact Kriemler shows that in rectangular columns

$$P' > P > P'',$$

that is to say, Euler's limit is nearer the correct value than v. Heim's. From a practical point of view, however, the difference between the three values is negligible.

Kriemler's result has been confirmed by Hasse (1905), Nussbaum (1907), and others. It follows that the effect of shear on the crippling load is small.

It is evident, therefore, that whatever may be the differences between the results of experiments and the Eulerian theory, these differences are due neither to the neglect of the direct compressive stress nor to the effect of shear. They are not due to incorrect theory, and all investigations in which it is sought to mend the theory by mathematical refinement result in modifications to Euler's formula which are negligible from a practical point of view.

Nevertheless, strict analysis has led to the elucidation of many obscure points. Lagrange very early remarked (1771) that the complete solution depends in general on the rectification of conic sections. The expression for the length of the bent elastic line is

$$l = \frac{1}{a\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{(\cos\theta - \cos\theta_a)}} \quad . \quad . \quad . \quad . \quad . \quad (403)$$

which is an elliptic integral.

Lagrange, Lamarle, Grashof, and others expressed $\theta$ in terms of $y$ the deflection, and expanded the denominator by the binomial theorem. The resulting series, as Lamarle showed, is very rapidly convergent. Bresse expanded the cosines, and later writers have actually used elliptic functions.

With the exception of Clausen's somewhat special analysis (1851), Clebsch (1862) appears to have been the first to apply elliptic functions to the general problem. A very complete demonstration of the connexion between these functions and the properties of the elastic curve was given by Greenhill in 1876; and Saalschütz (1880), Halpen (1884), Kriemler (1902), and others have determined the shape of a greatly deflected lamina under specified conditions of loading by their aid. These latter investigations, however, have no bearing on the strength of a practical column.

As has been seen, the strict analyses of Lagrange and others disposed of all apparent paradoxes, interruptions in the law of continuity, and like objections to the theory; but they showed that a number of deductions which appeared to follow from the simple theory were incorrect.

Instead of the deflection $y_0$ being an indefinite function of the load, it has a definite value for any value of W greater than P, and the shape of the column is absolutely determined by the value of the load. The column does not, in fact, pass into a state of unstable or neutral equilibrium (see p. 272).

The deflection curve is not a curve of sines, but bears the same relation to an ellipse that the sine curve does to the circle (Schneider, 1901). The true curve is, of course, the ***linteria*** of Bernouilli.

Clebsch calls the fact that the simple approximate theory gives the correct limit at which flexure begins a happy accident, and says that it is astonishing that the obviously absurd deductions from the simple theory should have been accepted without a search being made for the origin of the error (1862). Stress has recently been laid on this same point by Alexander (1912), for the same deductions are still made. It should be added, moreover, that Euler expressly limited his analysis to infinitely small deflections.

Meanwhile, Lamarle (1846) had carried the analysis one stage further. He showed that if the ideal column bend, the material in the most stressed fibre would immediately pass the elastic limit. Euler's limit load may therefore be looked upon not only as the load corresponding to the first deflection, but as the failure load of the specimen. Further, if the elastic limit load $(a.f_e)$ be less than P, the ideal column will fail by direct compression rather than by bending. This condition determines the value of $\frac{L}{\kappa}$, below which Euler's formula is inapplicable

$$\frac{L}{\kappa} > \sqrt{\frac{\pi^2}{s_e}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (404)$$

a limitation previously suggested by Young and Navier.

Lamarle's conclusions have been confirmed by Pearson (1886), who adopted the theory that permanent set is reached by lateral extension rather than by longitudinal compression. His limiting values for $\frac{L}{D}$ are, however, one half of those of Lamarle.

The latter's conclusion that the elastic limit is passed immediately the column begins to bend has also been confirmed by Schneider (1901), Gérard (1902), and Lorenz (1908).

Collignon (1889) put the matter in a different way. He showed that if W sensibly surpass P, the angular deviation $\theta_a$ very rapidly becomes considerable. Thus if

$$W = 1{\cdot}000152\ P \qquad \theta_a = 2^\circ$$
$$W = 1{\cdot}003\ P \qquad {}_a = 5^\circ$$
$$W = 1{\cdot}008\ P \qquad \theta_a = 10^\circ$$

The figures in the last two lines are given by Bredt (1894).

*Experimental Confirmation.*—In spite of the fact that it was the result of experiment which caused Euler's formula first to become suspect, it may be confidently affirmed that the issues of the Eulerian theory, limited in the manner proposed by Lamarle, have been completely demonstrated by experiment, ***provided that the experimental conditions conform to the theoretical assumptions.***

Thirty years before Euler published his memoir, Musschenbroek (1729) discovered experimentally that the strength of long columns varied inversely as the square of their lengths. Duleau (1820), by experiment on long, thin, wrought-iron specimens $\left(\frac{L}{D} = 87 \text{ to } 200\right)$, found that the ratio of the experimental to the theoretical crippling loads varied from 0·9 to 1·45, the mean being 1·16. Navier (1833), commenting on the experiments of his day, remarks that if precautions be taken to make the experimental conditions agree with Euler's hypotheses, the results are represented exactly by the formula. Hodgkinson (1840) found that for the longer specimens his index $n$ approached the value 2, thus agreeing with the theory. Winkler (1878), as the result of his examination of the Cincinnati Southern Railway experiments, concludes that Euler's formula represents the experimental results, when $\frac{L}{\kappa}$ is large, just as well as Rankine's formula. Burr (1884) considers that the results of Christie's experiments are accurately represented by Euler's formula when $\frac{L}{\kappa}$ is large. T. H. Johnson (1886), commenting on the shape of the mean $\left(f_r - \frac{L}{\kappa}\right)$ diagram, says: "That part of the line corresponding to the higher length ratios is a curve, the equation of which is Euler's formula."

From the experience of the earlier experimenters, therefore, it is fairly conclusive that for the larger values of $\frac{L}{\kappa}$ Euler's formula will represent the experimental results. Nevertheless, in the majority of cases, no very great care was taken to realize the theoretical assumptions, and to Bauschinger (1887) and Considère (1889) belongs the credit for the re-establishment of Euler's formula experimentally. The former's introduction of pointed ends led, in addition, to a long series of experiments by Tetmajer (1890), which may be said to have demonstrated finally that the ultimate strength of originally straight concentrically loaded specimens is represented by Euler's formula, provided that the load per unit area does not exceed the elastic limit of the material. This latter condition determines the validity limit of Euler's formula. Tetmajer gives as the value of the limit $\frac{L}{\kappa} = 105$ for mild steel specimens, and $\frac{L}{\kappa} = 112$

for wrought-iron specimens. These figures have, however, been called in question by several writers. Considère, as the result of his experiments, concluded that Euler's formula was only strictly true within very restricted limits, namely while the material is absolutely elastic. He would thus fix the validity limit, in the case of wrought-iron specimens, at $\frac{L}{\kappa} = 140$ to $150$, corresponding to a load of 9 to 10 kg/mm². In the case of steel specimens, the validity limit occurs at loads of from 12 to 20 kg/mm², depending on the hardness of the material. In this connexion Emperger's (1897) remark that Tetmajer's polygon of mean points for wrought-iron specimens first leaves the Euler curve at $\frac{L}{\kappa} = 180$, and finally at $\frac{L}{\kappa} = 150$, may be noted. On the other hand, Lilly (1908) fixes $\frac{L}{\kappa} = 120$ as the limit above which the modulus of elasticity governs the strength of columns and the ultimate strength closely approximates to Euler's crippling load. In the case of cast tool steel, the same author remarks that when $\frac{L}{\kappa}$ is greater than 70, no marked defect from the Eulerian curve occurs. For the open-hearth high-tensile steel with which he experimented, Kármán gives $\frac{L}{\kappa} \doteq 95$ as the validity limit for Euler's formula.

The limit evidently varies with the material, and a great deal depends on the accuracy of the experimenter. It may, however, be concluded that Euler's formula will give the ultimate strength of originally straight, concentrically loaded specimens in cases where the load per square unit does not exceed the elastic limit of the material. For practical purposes the validity limit, in the case of mild-steel specimens, is in the neighbourhood of $\frac{L}{\kappa} = 110$, although strictly speaking the absolute limit is probably higher still, say about $\frac{L}{\kappa} = 140$.

Judging from the shape of the deflection and stress curves for a column (Figs. 5 and 15), it appears that the probable value of the ultimate strength of long specimens is from 90 to 95 per cent. of Euler's value. For even with the smallest eccentricity or want of straightness, and no practical specimen can be perfect, the stress and deflection increase so rapidly when the load has reached this value that it may be considered to have failed. This would have the effect of increasing the value of $\frac{L}{\kappa}$ at the validity limit, and might account for the earlier falling away from Euler's curve observed.

Now a validity limit of $\frac{L}{\kappa} = 110$ in a position-fixed column corresponds to a validity limit of $\frac{L}{\kappa} = 220$ in a position- and direction-fixed column. The vast majority of practical columns of any importance are direction-fixed at the ends, and have a length ratio $\frac{L}{\kappa}$ less than 100. The objection to Euler's formula lies, therefore, not in its incorrectness, but in its utter inapplicability to ordinary practical cases.

Not only are the main theoretical deductions regarding the value of the

crippling load supported by experiment, but what may be termed the secondary consequences of the theory are likewise confirmed. Baker (1870) found that his specimens were stable when carrying 95 per cent. of their ultimate load, and Bauschinger (1887) when they were carrying $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of that load. Bauschinger measured the shape of some of his specimens and found the curvature to be approximately sinusoidal. Considère (1889) found that his specimens remained straight up to the point of failure, and that even with his delicate deflection indicator it was impossible to distinguish between the load which caused the first deflection and the failure load. Many of Tetmajer's (1890) specimens remained straight up to the failure point and then deflected suddenly, but they were chiefly specimens much too short for Euler's formula to apply. From his tables it appears that the longer specimens deflected gradually and the shorter ones suddenly, an indication of imperfect conditions. Kármán (1910) measured and plotted (Fig. 41) the deflections of long specimens near the failure point. His curves are those which might be expected from specimens under conditions approaching the ideal. The deflection is definite for each value of the load, but a small increase in the load produces a large increase in the deflection. This is the explanation of the phenomena noted by Houpt (1849), Baker (1870), and others, and advanced as a proof of a state of indifferent equilibrium, namely that with a given load the deflection may assume any value within limits.

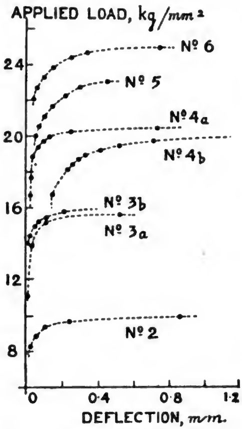



Fig. 41.—Deflection Curves for Long Steel Specimens (Kármán).

The evidence regarding the effect of direction-fixing is neither satisfactory nor conclusive. Very few experiments have been made with direction-fixed ends,* and no efforts appear to have been made to make certain that the direction-fixing was perfect; in fact, there is conclusive evidence that it was usually very imperfect. Nevertheless, Hodgkinson (1840) found that the resistance of long specimens with flanged ends was equal to that of specimens with round ends of one-half the length, as it theoretically should be.

The experimental testimony in favour of Euler's formula may be concluded with a reference to Sommerfeld's experimental analogy (1905), illustrative of the phenomenon of crippling.

To sum up:—

The Eulerian theory in its complete form may be looked upon as definitely established both mathematically and experimentally within the validity limits determined. Unfortunately those limits are outside ordinary practical conditions, which, together with the fact that the theory entirely neglects the imperfections inevitable in practice, renders it of little or no value for practical use. Nevertheless, as will be seen, Euler's crippling load is a measure of the *stability* of a column.

* For reasons which will appear later, the writer excludes specimens with flat ends from this category.

**The Eccentricity Formula.**—In spite of the experimental evidence quoted above, it cannot be denied that the *average* column either in practice or in the laboratory by no means behaves as, according to the Eulerian theory, it should. Instead of remaining straight up to the point of crippling and then failing suddenly and completely, it begins to deflect under the smallest load and fails before the load reaches Euler's limit. Hodgkinson (1840) remarks that flexure commences at very small loads, loads so small that the deflection cannot be measured. Bauschinger (1887) says that if sufficiently sensitive instruments be employed, deflections can be observed with the smallest loads. The testimony of experimenters is, in fact, unanimous on this point. Not only so, but instead of passing the elastic limit when the crippling load is reached, as according to Lamarle should be the case, the longer specimens are found to be absolutely uninjured after failure. Thus Christie found that all his specimens took permanent set when $\frac{L}{\kappa}$ was less than

231—flat ends,
174—flanged ends,
162—hinged ends,

and all recovered their original condition when $\frac{L}{\kappa}$ was greater than

370—flat ends,
312—flanged ends,
254—hinged ends.

Tetmajer and many others also have found that their longer specimens were uninjured after failure.

These and other departures from the behaviour to be expected according to Euler's theory are now generally admitted to be caused by imperfections in the conditions, eccentricity of loading, want of straightness, variations in the modulus of elasticity, and others of the same nature.

As already has been remarked, Young in 1807 was the first to include the effect of eccentricity of loading and initial curvature in his analysis. He considered the case of a rectangular column and reached formulæ which, in general symbols, reduce to:

Eccentrically loaded column,

$$\Delta = y_0 - \epsilon_2 = \epsilon_2 \left\{ \sec \frac{L}{2}\sqrt{\frac{W}{EI}} - 1 \right\} \quad . \quad . \quad . \quad . \quad (405)$$

Column with initial curvature,

$$y_0 = \Delta + \epsilon_1 = \frac{\epsilon_1}{1 - \frac{W}{P}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (406)$$

Navier (1833) gave a somewhat more general analysis for an eccentrically loaded column, but to Scheffler (1858) is due the complete solution of that problem. Scheffler took into account the direct stress due to the load, and showed that

$$y = \epsilon_2 \sec \frac{aL}{2} \cos ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (407)$$

and

$$W = \frac{f_c a}{1 + \frac{a v_2 \epsilon_2}{I} \sec \frac{aL}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (408)$$

where
$$a^2 = \frac{W}{I\left(E - \frac{W}{a}\right)},$$

and every correct mathematical analysis has resulted in a confirmation of his formulæ.

The more general solution to the problem of a column with initial curvature was given by Ayrton and Perry (1886), and partly also by Fidler (1886). A more exact analysis, taking into account the direct stress due to the load, will be found on p. 48. Ayrton and Perry's equations are

$$y_0 = \frac{\epsilon_1}{1 - \frac{W}{P}} \qquad (409)$$

and
$$\frac{f_c - f_a}{f_a} = \frac{v_2}{\kappa^2} \cdot \frac{\epsilon_1}{1 - \frac{W}{P}} \qquad (410)$$

The shape of the above expression for $y_0$ led Bauschinger (1887) to formulate a new definition for Euler's crippling load. Instead of being the smallest load which could produce deflection, it is the limiting value of the load under which the already existing deflection becomes infinitely great and the resistance of the column is overcome. Föppl (1897) says the same thing in a different way. He points out that when $W = P$, the deflection becomes infinitely great, whatever be the value of $\epsilon_1$. Hence the value of the crippling load, within known limits, is independent of the value of $\epsilon_1$, provided that $\epsilon_1$ be small; and on this independence the applicability of Euler's formula depends.

*Experimental Confirmation.* The experimental confirmation of the eccentricity formula is perhaps not so complete as in the case of Euler's formula. Nevertheless, within the elastic limit, both the deflections and stresses obtained by experiment can be interpreted with exactness by the formula.

Meyer, by his experiments (Tetmajer, 1896), may be said to have established the formula directly. Tetmajer's own experiments (1890) are an indirect confirmation. Ayrton and Perry (1886) showed that it was possible by choosing a suitable value for $\epsilon$, the equivalent initial curvature, to give a close interpretation of the deflection curves of Hodgkinson; and Moncrieff (1901), by assuming an initial curvature combined with an eccentricity of loading, showed that both Hodgkinson's and Christie's deflection curves could be represented with a considerable degree of accuracy.

Nevertheless, a direct calculation of the eccentricity from the observed deflections leads to a very irregular set of values for $\epsilon_1$ and $\epsilon_2$. The present writer spent some time in trying to differentiate between the effect of original curvature and eccentricity of loading as displayed in experimental deflection curves. These attempts resulted in widely differing values for $\epsilon_1$ and $\epsilon_2$, and he came to the conclusion that the unavoidable errors of observation in the very small deflection readings far exceeded the differences between the effects of initial deflection and eccentricity of loading. Further, when the deflection becomes large, there appears to be a growth in the value of the eccentricity, due probably to local permanent sets.

These two causes, considerable errors in the small *difference* between two inexact experimental observations and variation in the eccentricity during the course of the experiment, appear to be sufficient to explain the irregular values

of the eccentricity as obtained by direct application of the formula; and its successful indirect application as evidenced by Ayrton and Perry and Moncrieff may therefore be looked upon as experimental proof of its correctness.

It may be well to notice here the view, which appears to have originated with Grashof, that the column is an exception to the ordinary theory of elasticity. No proof of this is possible except by experiment, and there appears to be no experimental justification for such a supposition. It has been seen that the consequences of the Eulerian theory are borne out by experiment, when the experimental and the theoretical conditions agree. Meyer, Dupuy (1896), and others have shown that the measured stress in a specimen agrees with that calculated by the ordinary theory; which, coupled with the fact that the observed deflections can be accounted for by the eccentricity formula, is a demonstration that the ordinary theory of elasticity is sufficient to explain completely the behaviour of a column.

With regard to failure loads, since these imply that the elastic limit has been overstepped, the eccentricity formula must cease to hold. Nevertheless, Marston (1898) has shown that, by assuming a suitable value for $\frac{\epsilon_2 v_2}{\kappa^2}$, Tetmajer's results can be represented with a considerable degree of accuracy. Prichard (Lilly, 1913) comes to a similar conclusion regarding Lilly's experiments on solid cylindrical specimens, and Fidler (1886) also uses a variation of the same formula to represent the ultimate strength.

The eccentricity formula may be said, therefore, to be firmly established both theoretically and experimentally. If it be modified to include the effect of initial curvature in addition to the eccentricity of loading, it will represent the actual behaviour of the column better than any other formula, provided that the elastic limit be not exceeded.

The formula is, however, somewhat inconvenient to apply in practice. So far as mere arithmetical difficulties are concerned, this may be overcome by curves and tables (Smith, 1878 and 1887), or approximations may be used.

More important is the difficulty of determining the values of $\epsilon_1$ and $\epsilon_2$ in practical cases, complicated as it is by the uncertainty regarding end conditions. Since, however, eccentricity of loading, initial curvature, and imperfections in the end conditions are the prime sources of weakness in all columns, this difficulty is inherent in all column formulæ, and the artifice of cloaking real ignorance by the introduction of constants is no solution to the difficulty. This difficulty is, in fact, the essential difficulty of the column problem, and is not peculiar to this formula.

The objections raised by Emperger (1897) and others appear groundless in view of the experimental confirmation, nor does there seem any reason why application of the formula should be restricted to those limits within which Euler's formula is valid.

To one feature of the deflection curve the author would direct attention. It will be seen from Fig. 5 that while $\frac{W}{P}$ is less than $\frac{1}{4}$, which it always is in practical cases, the deflection curves for both eccentrically loaded and initially curved specimens are very nearly straight lines. If, therefore, these curves be replaced by the mean straight line as suggested in Part II, p. 43, the following simple practical formula results:

$$f_c = f_a\left\{1 + \frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\left(1 + \frac{3W}{2P}\right)\right\} \quad . \quad . \quad . \quad . \quad (411)$$

If the value of $\epsilon_1$ and $\epsilon_2$ be chosen so as to include the effect of variation in the modulus of elasticity, this single expression will enable all the known imperfections in position-fixed columns to be taken into account, and within the specified limits can be safely applied to such columns.

**Variations of the Eulerian Analysis.**—In addition to the ordinary analysis of Euler, and the many "strict" demonstrations of the formula which have been given, a number of approximations and variations in the method of proof have from time to time appeared. Some of these are of interest.

Semi-graphical demonstrations based chiefly on Möhr's theorem have been given by Fidler (1886), Land (1896), Schüle (1899), and others. Graphical methods of obtaining the shape of the bent elastic line have been given by Bredt (1894), Duclout (1896), and Vianello (1898). Duclout's application of the funicular polygon might be usefully extended.

Körte (1886), Bredt (1886 and 1894), and others have assumed that the curvature of a column is uniform, obtaining thus simple formulæ in which the effects of the various imperfections can be readily included. From a table given by Bredt it would appear, however, that if these formulæ be applied to uniform columns, for a given value of $\epsilon_2$ the value of the deflection obtained may be 20 per cent. too small.

Jasinski (1894), Wittenbauer (1902), Zimmermann (1905, 1907, and 1909), and Prichard (1909) have given generalized forms of Euler's formula, including in their analysis various secondary effects in addition to that of the longitudinal load.

Stoney (1864), Moncrieff (1901), and others have assumed the deflection curve to be parabolic in shape instead of sinusoidal. Cain (Moncrieff, 1901) remarks that this assumption gives a closer approximation for small values of $\frac{L}{\kappa}$ than is obtained by expanding the secant and neglecting higher powers of the angle than the second.

Many writers have assumed the deflection to be produced by an equivalent transverse load, reaching thereby an approximate value for the crippling load. Possibly the work of Chaudy (1890) is the most complete exposition of this method of attack. Vierendeel's assumption (1904) that the transverse loads are virtual forces produced by the lattice bracing of the column is a novel development.

The analysis ascribed to John Neville (Neville, 1902), which leads to Lamarle's formula, is of interest from a mathematical point of view.

**The Rankine-Gordon Formula.**—As has been seen, one of the objections raised to Euler's formula was that the direct compressive stress had been neglected. The earliest attempts to remedy this defect resulted in the formula which is known in Great Britain as the Rankine-Gordon formula, and in Germany as the Schwarz or Schwarz-Rankine formula.

Tredgold (1822) appears to have been the original author, but his formula does not appear to have come into general use until Gordon adapted it to represent the results of Hodgkinson's experiments. This must have been later than 1840, though the actual date of Gordon's work cannot be ascertained.*

* The author obtained no response to the following letter inserted in *Engineering*.

"The Editor, *Engineering*, London.

"As a matter of historical interest, would you allow me to trespass on your space to inquire if any of your readers could give me particulars as to where and when Gordon

Tredgold's analysis applies to rectangular cross sections only (he gives a modification of the formula to be used for circular cross sections), and Gordon's formula

$$f_a = \frac{c_1}{1 + c^2\left(\frac{L}{D}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (412)$$

depends on the ratio $\left(\frac{L}{D}\right)^2$. Later on, Rankine transposed the formula, altering the ratio to $\left(\frac{L}{\kappa}\right)^2$:

$$f_a = \frac{f_c}{1 + c_2\left(\frac{L}{\kappa}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (413)$$

The formula appears in this form in *Useful Rules and Tables*, London, 1866. In the first edition of *Applied Mechanics*, London, 1858, Gordon's formula only is given.

Meanwhile, Schwarz (1854) and Laissle and Schübler (1857) had given an analysis resulting in the same formula, except that the ratio $\left(\frac{L}{\kappa}\right)^2$ is given as $\frac{aL^2}{I}$. The formula should therefore be called the Tredgold-Schwarz formula.

It has been the subject of almost as much controversy as Euler's formula. On the one hand its claims as a rational formula have been attacked, on the other its merits as an empirical expression of experimental results have been decried.

Since the analysis by which the formula has been derived is founded on the elastic theory, it is desirable to see to what extent it may lay claim to be rational.

Its ultimate basis is undoubtedly the statement that the maximum compressive stress in the column is the sum of that due to direct compression and that due to bending:

$$f_c = f_a + f_b$$

Now, if the centre of resistance of the cross section in which the maximum stress occurs lie at a distance $y_0$ from the load line,

$$f_b = \frac{W y_0 v_2}{I}$$

and

$$f_c = f_a + f_a \frac{y_0 v_2}{\kappa^2}$$

or

$$f_a = \frac{f_c}{1 + \frac{y_0 v_2}{\kappa^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (414)$$

It may be observed, in passing, that this formula will hold for a column under any conditions, from eccentrically loaded to direction-fixed, provided that the proper value be given to $y_0$.

published his well-known column formula. I am, of course, aware of Rankine's remark to the effect that Gordon revised Tredgold's formula and determined the constants from Hodgkinson's experiments, but Gordon's own work I am unable to find."

To the analysis so far no objection can be raised. In a given case $f_a$ or $f_c$, $v_2$ and $\kappa$ can all be determined, but unfortunately $y_0$ is indeterminate, and cannot be expressed in terms of the known factors of the problem. This difficulty is encountered in every "proof" of the formula from Tredgold's analysis onward, and the numerous variations which have been suggested are merely endeavours to avoid this obstacle. It may be well to lay some stress on this point. If a quantity depend on factors which are not merely unknown, but which cannot from the nature of the case be determined, then it cannot be expressed in terms of known factors, and by no amount of mathematical ingenuity can it be evaluated. Now the value of $y_0$ depends on the magnitude of the initial curvature, the eccentricity of loading, and the imperfections in the end conditions, and in general these are indeterminate. Hence $y_0$ cannot be expressed in terms of known factors. Much effort has been wasted in attempting to evade this elementary proposition. In this difficulty lies the weakness of the Rankine-Gordon formula: it is the essential difficulty in the column problem, and was met before in the Eccentricity formula. It may be well to examine some of the efforts made to overcome it.

Tredgold (1822) assumed the curvature to be circular, and found that

$$y_0 = \epsilon_2 + \frac{L^2 s_b}{4D}.$$

He further assumed that when $f_c = f_e$, $s_b = s_e$, thus eliminating the unknown factor $s_b$. Ritter (1865) adopted the same device, arguing that since $s_b$ must be less than $s_e$ when $f_c = f_e$, the error introduced is on the side of safety. Schwarz (1854) replaced the value of E in Euler's formula by $\frac{f_e}{s_e}$. Hence

$$f_e = \frac{W s_e L^2}{\pi^2 I}.$$

He assumed then that $f_b = f_e$, and therefore

$$f_c = \frac{W}{a} + \frac{W s_e L^2}{\pi^2 I},$$

or

$$W = \frac{f_c a}{1 + \frac{s_e}{\pi^2} \cdot \frac{aL^2}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (415)$$

Laissle and Schübler (1857) in their earlier editions treat $s_b$ as a constant. In the later editions they assume that since $y_0$ increases rapidly with increasing length, and diminishes with increase in the dimensions of the cross section, therefore $y_0$ varies as $\frac{L^2}{v_2}$, and the formula becomes

$$W = \frac{f_c a}{1 + c \cdot \frac{aL^2}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (416)$$

Rankine likewise (1858), and many other writers, by analogy with transverse bending, assume that $y_0 = c_2 \frac{L^2}{D}$, thus introducing an unknown constant $c_2$ in place of $y_0$. Now $y_0$ is a function of W or $f_a$. Hence $c_2$ is a function of W, and cannot be a constant. This has been pointed out by many writers. Smith

(1878) remarks that the above assumption leads to the value $f_b = c_2 W \frac{L^2}{I}$, whereas actually $y_0 \propto f_b \frac{L^2}{D}$, and therefore $f_b \propto f_b W \frac{L^2}{I}$, which gives no information regarding the value of $f_b$. Tetmajer (1896), from his experimental results, calculated the value of $c_2$ both for different experiments and for different values of the load in the same experiment. He found that $c_2$ was far from constant. Tetmajer's method is open, however, to objections. Körte (1886) and others have made similar criticisms of the formula.

It may be argued that even if $c_2$ vary through the experiment, for the point of failure it is constant. This is far from being proved; but even if it be the case, the formula has then ceased to be rational and has become empirical. Its value from this point of view will be considered later.

Several writers have introduced a connexion with Euler's formula. Schwarz's analysis is a case in point. Grashof (1866) sought an expression which would reduce to $W = f_c a$ when $\frac{L}{\kappa} = 0$, and to $W = P$ when $\frac{L}{\kappa} = \infty$.

A suitable expression is

$$W = \frac{R_0 \times P}{R_0 + P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (417)$$

where $R_0 = f_c a$. This is a modification of the Rankine-Gordon formula, to which it will reduce. Hodgkinson (1840) proposed a somewhat similar expression, but replaced P by the experimental crippling load for long specimens. Grashof's formula has also been suggested by later writers. Merriman (1882) substituted $(f_c - f_a)$ for $f_b$ in the denominator, and remarks that by solving for $f_a$ Euler's formula

$$f_a = c\,E\,\frac{\kappa^2}{L^2}$$

is obtained. Burr (1882) writes Euler's formula in the form

$$f_b = f_b \times f_a \frac{L^2}{4\pi^2 E \kappa^2}.$$

Hence

$$f_a = \frac{f_c}{1 + \frac{f_b}{4\pi^2 E} \cdot \frac{L^2}{\kappa^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (418)$$

The constant $c_2$ in the Rankine-Gordon formula, which is equal to $\frac{f_b}{4\pi^2 E}$, is evidently not a constant, and the equation "is simply a redundant form of Euler's formula."

In this connexion it is of interest to compare the Lamarle and Schwarz formulæ:

$$W = \frac{Ea}{1 + \frac{L^2 a}{\pi^2 I}} = \frac{f_e a}{s_e + s_e \frac{L^2 a}{\pi^2 I}} \quad . \quad . \quad . \quad . \quad (\text{Lamarle})$$

$$W = \frac{f_e a}{1 + s_e \frac{L^2 a}{\pi^2 I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{Schwarz})$$

Pearson (1886) deduces a formula of the Rankine-Gordon type from Scheffler's formula by assuming that

$$\epsilon_2 = \epsilon_0 \left( \frac{y_0 - \epsilon_2}{y_0} \right),$$

where $\epsilon_0$ is the value of $\epsilon_2$ for a very long column, and Navier (1833) obtained an expression very similar to the Rankine-Gordon formula from the eccentricity formula.

Lilly (1908) has ingeniously proposed to evaluate $f_b$ by a continued expansion of the denominator.

Several writers have introduced the factor of safety into their analysis in order to overcome the fundamental difficulty; for example, Winkler (1867) and Pilgrim (1904). The method by which Barth (1898) eliminates the values of the unknown eccentricities is much the same.

Crehore (1879) reduces the moment of stiffness in the ratio $\frac{f_b}{f_c}$, and thus deduces the Rankine-Gordon formula from Euler's. T. H. Johnson (1888) and Lilly (1904) have divided the area of the column into two parts, the one resisting the direct stress and the other resisting the bending moment, obtaining by this means the Rankine-Gordon formula; but this is an objectionable device.

It will be evident from the above, which by no means includes all the attempts, that no ingenuity legitimate or illegitimate has been spared in order to overcome the inherent difficulty of the problem. The various evasions, embodied in the formula in the form of the many variations in the constants proposed, advance us not one step towards the solution; and in the absence of definite information regarding the unknown factors there seems no reason why, instead of assuming constants, the deflection $y_0$ should not be assumed directly. It is evidently $y_0$ which is assumed whatever be the mathematical shape into which the formula is thrown, and its direct assumption has some advantages. This, in effect, is what several modern German writers propose to do.

The problem is worked backwards in an editorial in *Engineering News* (1907). There the value of $y_0$ is calculated from the more common empirical formulæ. The results, as might be expected, differ widely both as to actual values and also in regard to the factors on which $y_0$ depends.

To sum up, the basis of the Rankine-Gordon formula is rational; it is, in fact, the fundamental condition on which all stress formulæ for columns are based, namely that the total stress is composed of the sum of that due to direct compression and that due to bending. The formula is not a solution of the inherent difficulty in all column problems, viz. that the actual conditions are unknown, and attempts to overcome this difficulty by its use are a priori doomed to failure. The formula loses its rational character and becomes empirical when constants are introduced in place of $y_0$.

**The Eccentricity Form of the Rankine-Gordon Formula.**—In addition to the various shapes which have been given to the Rankine-Gordon formula for columns with position-fixed ends, several authors have introduced terms into the denominator to allow for eccentricity of loading and other imperfections, or transverse bending.

As has been pointed out, the formula (414),

$$f_a = \frac{f_c}{1 + \frac{y_0 v_2}{\kappa^2}},$$

holds for eccentricity of loading or any other condition of loading provided that the correct value is given to $y_0$. No extra terms, therefore, are necessary. It may be observed, in fact, that Tredgold's analysis was given for an eccentrically loaded column, and the formula for the concentrically loaded followed as a special case. Nevertheless, if the column have a known eccentricity (or a transverse load as the case may be), that portion of $f_b$ due to the known factors may be evaluated, leaving the unknown factors still in the form $c_2\left(\frac{L}{\kappa}\right)^2$. This is what Tredgold (1822), Cain (1887), Pullen (1896), and many others have done. A word of caution with regard to the method is, however, necessary. If the eccentricity be known, $y_0$ may be replaced by its equivalent $(\epsilon_2 + \Delta)$, and

$$f_c = f_a\left(1 + \frac{y_0 v_2}{\kappa^2}\right) = f_a\left(1 + \frac{\epsilon_2 v_2}{\kappa^2} + \frac{\Delta v_2}{\kappa^2}\right),$$

and the formula becomes

$$f_a = \frac{f_c}{1 + \frac{\epsilon_2 v_2}{\kappa^2} + \frac{\Delta v_2}{\kappa^2}} = \frac{f_c}{1 + \frac{\epsilon_2 v_2}{\kappa^2} + c_2\left(\frac{L}{\kappa}\right)^2} \quad . \quad . \quad . \quad (419)$$

where $c_2 = \frac{\Delta v_2}{L^2}$.

Now if it be legitimate to replace $y_0$ by a constant, it is just as legitimate to replace $\Delta$ by a constant. But since $\Delta$ is a function of $\epsilon_2$, $c_2$ in the above formula is not the same as $c_2$ in the usual Rankine-Gordon formula, as so many writers have supposed, and the fact that the formula reduces to the Rankine-Gordon when $\epsilon_2$ is zero is not a proof of the identity of the constants, for $c_2$ is a function of $\epsilon_2$. It is obvious, in fact, that $c_2$ must be greater if the eccentricity $\epsilon_2$ be greater, for $\Delta$ will be greater. Ostenfeld (1898), in his estimation of the values of $c_2$, finds that for concentrically loaded specimens $c_2 = 0{\cdot}000093$ and for eccentrically loaded specimens $c_2 = 0{\cdot}00018$. If, in addition to the known eccentricity, there be also an unknown eccentricity, $c_2$ in the eccentricity form of the formula must include the effect of this, and hence in general will be a complicated function of both the known eccentricity and the imperfections in the conditions.

**Imperfections in Columns.**—It is now generally admitted that the chief sources of weakness in a column are the small imperfections in the physical conditions under which it exists, that these imperfections are inevitable, and that the real cause of the supposed divergence between theory and practice is the entire neglect or incorrect estimation of the magnitude of these imperfections in the usually accepted theory.

As has been pointed out, the difficulty of determining the magnitudes of these imperfections is the essential difficulty of the column problem, for until they are known it is impossible to predict the behaviour of the column. Having, however, determined their values, the column may be designed by the aid of the formulæ already considered.

It may be laid down at the outset that to determine these magnitudes with exactness for any particular case is impossible. All that can be done is to determine limits within which the values of the imperfections will in all

probability lie. To this end it will be well to consider the nature of those more commonly occurring. They may be divided into three groups:

| Material. | Manufacture. | Conditions of Loading. |
|---|---|---|
| Want of Homogeneity.<br>Variations in the Modulus of Elasticity.<br>Initial Stresses.<br>Local Permanent Sets.<br>Effect of Past History.<br>Local Defects.<br>Flaws.<br>Knots and Shakes in Timber. | Initial Curvature.<br>Stresses set up during Manufacture.<br>Effect of Cold Straightening.<br>Inequality of Areas.<br>Creeping due to Riveting.<br>Annealing.<br>Imperfect Castings. | Eccentricity of Loading.<br>Imperfect End Conditions. |

These items may be further regrouped under three new headings depending on their effect on the strength of the column:

| Eccentricity of Loading. | Initial Curvature. | Reduction in Strength of Material. |
|---|---|---|
| Eccentricity of the Load.<br>Variations in the Modulus of Elasticity.<br>Inequality of Areas.<br>Imperfect Castings. | Initial Curvature.<br>Variations in the Modulus of Elasticity.<br>Creeping due to Riveting. | Effect of Past History.<br>Effect of Cold Straightening.<br>Stresses set up during Manufacture.<br>Initial Stresses.<br>Local Permanent Sets.<br>Annealing.<br>Flaws and Local Defects.<br>Imperfect Castings.<br>Knots and Shakes in Timber. |

Thus variations in the modulus of elasticity may be equivalent to eccentricity of loading or initial curvature, or to the combination of both. Variation in the area of the rolled sections forming a column produces an equivalent eccentricity of loading. An eccentric core in a cast column has a like effect.

The effects of the end conditions and imperfections in the same need special consideration. In general they may be looked upon as producing either a positive or negative eccentricity of loading.

Broadly speaking, therefore, it may be said that the imperfections in a column produce three distinct effects:

Eccentricity of loading.
Initial curvature.
Reduction in the strength of the material.

Eccentricity of Loading.—Of these, eccentricity of loading is the one most commonly taken into account. Its power to weaken a position-fixed column is well known, and the testimony of experimenters is practically unanimous on this point. Further, the magnitude of the eccentricity need by no means be large in order to have a considerable effect on the strength of the column. Christie (1884) remarks that minute changes in the centre of pressure made great alteration in the strength of the column, and sometimes by moving the specimen apparently slightly out of centre the resistance was vastly increased.

Nevertheless, a small error in the estimate of the magnitude of the eccentricity is not of great moment, for a slight increase in an already existing eccentricity does not greatly increase its effect. The results of Considère show that the resistance falls rapidly when the relative eccentricity $\left(\frac{\epsilon_2}{\kappa}\right)$ is very small, but the rate of diminution of strength is much reduced as the eccentricity grows in magnitude. Ayrton and Perry (1886) would explain the somewhat curious phenomenon noted by Hodgkinson, namely that a small eccentricity in the core of hollow cast-iron columns did not much affect their strength, on these grounds. Assuming that imperfections exist in all specimens, an additional eccentricity due to the core would not produce a further large reduction in strength.

However this may be, the fact that a small error in the estimate of the eccentricity is unimportant is of consequence in view of the difficulty in determining its exact value. It is almost needless to say that the estimates have been many and differ widely.

In general, two courses are open. Firstly, the probable magnitude of the eccentricity due to the causes enumerated may be estimated and the sum taken as the total eccentricity. Here a difficulty arises in that a column is unlikely to suffer from all possible diseases at once, nor is the direction of all the eccentricities likely to coincide. Smith (1878) suggests an application of the theory of probabilities, and would take $\frac{7}{11}$ of this total eccentricity as the value of $\epsilon_2$, and assume it to act in the worst possible direction.

Secondly, the total eccentricity may be determined from experimental data in particular cases and the mean or maximum value of a series taken as the probable value in practice. This second method, supposing it to be possible to determine the eccentricity in a sufficiently large number of cases, has the advantage that it determines at once the probable eccentricity in any one particular direction. It has the great drawback that the eccentricity so determined is the eccentricity likely in a test specimen, which is totally different from, and bears no relation to, the eccentricity likely in practice, where the conditions are absolutely different.

In applying the first method, the most difficult factor to estimate is the magnitude of the error in centering the load. Tredgold boldly argued that the points of application of the load should be assumed to lie on the contour of the end cross sections. In most columns intended to be concentrically loaded this gives the maximum possible value of $\epsilon_2$, and is only likely, as will be shown, at the beginning of the experimental history of a column with flat ends.

In Tetmajer's experiments (1896), where the specimens were much better centered than is probable in practice, $\epsilon_2$ varied from 0·003 to 0·102 cm. when the load was small, increasing to a maximum of 0·708 cm. when the load increased. Morris (1911) found the mean eccentricity in Buchanan's experiments to be about $\frac{1}{4}$ in., and Kirsch (1905) suggests 10 mm. These figures, however, are equivalent eccentricities rather than actual errors in centering. Further, it is not improbable that the error would increase with the size of the specimen. Most writers have assumed the eccentricity to be a function of the radius of gyration, and there appears to be a certain concensus of opinion in favour of a value $\frac{\kappa}{10}$; equal, according to Jensen (1908), to that in the worst of Tetmajer's specimens. The author, for reasons discussed later, considers that the error in centering is more likely to vary with the length of the specimen than its

radius of gyration, and would suggest the value $\frac{L}{1000}$. This agrees with the estimate $\frac{\kappa}{10}$ when $\frac{L}{\kappa} = 100$. It has the advantage that it is easy to calculate, since L is always known.

A better estimate may be made regarding the eccentricity due to variations in the modulus of elasticity. As has been seen in Part II, if one side of the column have a lower modulus of elasticity than the other, the variation is equivalent to a combination of eccentricity of loading and initial deflection. Young (1807) first pointed out the effect of such a variation, and Fidler (1886) determined a value for the equivalent eccentricity. According to Fidler's investigation

$$\epsilon_2 = \frac{D}{2} \cdot \frac{E_1 - E_2}{E_1 + E_2} = \kappa \frac{E_1 - E_2}{E_1 + E_2}$$

He assumes the maximum and minimum values of E for wrought iron to be $E_1 = 29,000,000$ lb. sq. in., and $E_2 = 23,000,000$ lb. sq. in., giving a value of $\frac{E_1 - E_2}{E_1 + E_2} = 0 \cdot 117$, so that the greatest possible value for $\epsilon_2$ is $\frac{1}{9}\kappa$, or more strictly $\frac{D}{18}$. This value of $\epsilon_2$ is apparently intended by Fidler to represent either eccentricity of loading, initial deflection, permanent set, irregularity of the modulus, or a combination of them all. It is determined solely from the variation in the modulus.

Neville (1902), by making the assumption that the modulus varied uniformly from one side of the cross section to the other, found that for a symmetrical cross section

$$\epsilon_2 = \frac{2\kappa^2}{D} \cdot \frac{E_1 - E_2}{E_1 + E_2}$$

Like Fidler, he takes $\frac{1}{9}$ as the value of $\frac{E_1 - E_2}{E_1 + E_2}$

Bredt's assumption (1894) of a variation in the modulus which will leave the line of resistance coincident with the central axis appears too artificial to be of practical importance.

From the analysis in Part II, which is more general and complete than any hitherto given, it would appear that the correct expression for the eccentricity equivalent to a variation in the modulus similar to that assumed by Fidler is

$$\epsilon_2 = e \frac{a_1 \bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where

$$e = \frac{E_1 - E_2}{E_a}$$

A calculation of the value of $e$ from the results of experiments by a number of different writers is given in the accompanying table. As will be seen, the values for $e$ vary considerably. From the tensile tests it would appear that the value of $e$ is small, usually less than $0 \cdot 1$; for the compression tests it is much greater and more variable. It would appear, in fact, from the figures collected, that the average value of the modulus of elasticity $E_a$ is less in compression

than in tension, and that the difference ($E_1 - E_2$) is enormously greater in the compression tests, with the result that the value of $e$ rises, in Christie's experiments, to values of 0·565, 0·457, and 0·512 for wrought iron, mild steel, and hard steel specimens respectively. Part of this variation is doubtless due to the deflection which inevitably accompanies compression tests. The side of the specimen on which measurements are made may become either convex or concave, and therefore *both* maximum and minimum values for E might be expected to show larger variations from the average than is the case in tension specimens, and this may be seen in the table. It is not so easy, however, to explain the large variations in E recorded by Christie in his bending experiments.

VARIATION IN THE MODULUS OF ELASTICITY

| Experiments. | $E_1$ | $E_a$ | $E_2$ | $E_1 - E_2$ | $e$ |
|---|---|---|---|---|---|
| CHRISTIE (1884). | | | | | |
| Wrought Iron: | | | | | |
| Tension | 29,400,000 | 28,416,000 | 27,270,000 | 2,130,000 | ·075 |
| Compression | 35,300,000 | 27,093,000 | 20,000,000 | 15,300,000 | ·565 |
| Bending | 33,631,000 | 27,663,000 | 19,164,000 | 14,467,000 | ·523 |
| Mild Steel (0·12 C): | | | | | |
| Tension | 32,780,000 | 30,135,000 | 27,030,000 | 5,750,000 | ·191 |
| Compression | 24,490,000 | 20,478,000 | 15,132,000 | 9,358,000 | ·457 |
| Bending | 32,930,000 | 29,758,000 | 18,765,000 | 14,165,000 | ·476 |
| Hard Steel (0·36 C): | | | | | |
| Tension | 30,000,000 | 29,280,000 | 28,570,000 | 1,430,000 | ·049 |
| Compression | 30,770,000 | 24,570,000 | 18,182,000 | 12,588,000 | ·512 |
| Bending | 28,037,000 | 27,163,000 | 27,008,000 | 1,029,000 | ·038 |
| TETMAJER (1890). | | | | | |
| Wrought Iron in Tension: | | | | | |
| Angle | 20,270 | 19,890 | 19,540 | 730 | ·037 |
| Tee | 19,870 | 19,520 | 19,270 | 600 | ·031 |
| Channel | 20,860 | 19,760 | 19,330 | 1,530 | ·078 |
| Mild Steel in Tension: | | | | | |
| Angle | 21,840 | 21,420 | 20,730 | 1,110 | ·052 |
| Tee | 22,550 | 21,760 | 20,850 | 1,700 | ·078 |
| BAUSCHINGER (1887). | | | | | |
| Wrought Iron: | | | | | |
| Compression | 2,270,000 | 2,048,500 | 1,810,000 | 460,000 | ·225 |
| WATERTOWN ARSENAL (1908–12). | | | | | |
| Mild-Steel Columns: | | | | | |
| 18 Experiments | 37,870,000 | 30,960,000 | 26,960,000 | 10,910,000 | ·353 |
| Three extreme cases neglected: | | | | | |
| 15 Experiments | 33,560,000 | 30,630,000 | 28,490,000 | 5,070,000 | ·165 |
| FIDLER (1886). | | | | | |
| Wrought Iron: | | | | | |
| (Assumed Values) | 29,000,000 | 26,000,000 | 23,000,000 | 6,000,000 | ·230 |

Units:—Tetmajer's . . . . . . . . . . kg/mm²
Bauschinger's . . . . . . . . . . kg/cm²
Others . . . . . . . . . . lb. sq. in.

On the other hand, the carefully carried out experiments of Bauschinger on short wrought-iron specimens exhibit a variation equivalent to a value

$e = 0{\cdot}225$. The results of 18 experiments on long mild-steel columns, picked at random from the Watertown Arsenal Reports, with all types of end bearings, show a value $e = 0{\cdot}353$. If the three more extreme cases be neglected on the ground that such variation is due to bending, a fairly compact group of experimental results remains, from which a value $e = 0{\cdot}165$ was obtained.

It would appear, therefore, that a value $e = 0{\cdot}2$ is a reasonable and moderate estimate of the probable variation in the modulus of elasticity. Fidler's assumed variation in the modulus gives $e = 0{\cdot}23$.

In a built-up column consisting of two separate flanges, the designer should expect such a difference in the modulus of elasticity of the two flanges. It would appear that in solid columns also such a variation would not be very exceptional.

The case of a Krupp shaft is quoted by Fidler.* From the results of tests made by Kirkaldy, the strain in sixteen tests under direct stress varied from 0·032 to 0·047, corresponding to a value $e = 0{\cdot}380$; and similarly in six tests in bending, a variation of 0·038 to 0·049, corresponding to a value $e = 0{\cdot}253$, was observed. Tetmajer made tensile tests of the variation in the modulus of elasticity in the webs and flanges of channels. In all cases the modulus was greater in the web than in the flanges, which suggests that the manner of rolling such shapes affects the value of E. The value of $e$ in this case was 0·048.

In a solid column, therefore, the eccentricity of loading due to variations in the modulus of elasticity may be taken as

$$\epsilon_2 = \frac{1}{5}\,\frac{a_1\bar{v}_1}{a},$$

and in a built-up column in which the two flanges are equal in area

$$\epsilon_2 = \frac{h}{20} = \frac{D}{20} \text{ [see equation (73)].}$$

In recent years, more particularly in connection with the strength of built-up columns, it has been pointed out † that the unavoidable variation in the sectional area of the two flanges of a column is equivalent to a considerable eccentricity of loading.

Rolling mills claim as a rolling margin a tolerance of $2\frac{1}{2}$ per cent., and in some cases of as much as 5 per cent., above or below the specified weight of the rolled material. It is therefore possible that in a built-up column of which the two flanges are composed of rolled sections, the area of one may be $2\frac{1}{2}$ per cent. greater, and that of the other $2\frac{1}{2}$ per cent. less than that specified. This means that the centre of resistance of the cross section will be shifted $1\frac{1}{4}$ per cent. of the distance between the centres of area of the flanges toward the heavier flange, causing an equivalent eccentricity

$$\epsilon_2 = \frac{h}{80}, \text{ or approximately } = \frac{D}{80}.$$

That such variations in the area of the cross section do occur has often been noted by experimenters, and even if eccentricity of loading be not set up, the difference of area may make a considerable difference to the ultimate strength. Thus Bauschinger (1887) found that the weight per metre run

* *A Practical Treatise on Bridge-Construction*, p. 168.

† See, for example, Basquin (1913).

varied even in specimens cut from the same bar, showing that slight variations in the cross section existed. To obtain the elements of the section exactly, he transferred the shapes to paper, and calculated for each section the required information.

Tetmajer (Emperger, 1897) remarks that it was not until the exact sizes of his specimens were taken into account that the agreement between Euler's formula and the experimental crippling load became apparent. Emperger (1908) records a difference in area of 10 per cent. between the nominal and exact size of his specimens. In the experiments made for the new Quebec Bridge (1910) it was found that the members varied from 1 per cent. light to 2 per cent. heavy, and in the nickel steel experiments (1914) it is recorded that some of the specimens were 1·6 per cent. light.

It would therefore appear that a variation of $\pm 2\frac{1}{2}$ per cent. between the flanges of a built-up section is at least a practical possibility, and a variation of one-half this amount, equivalent to an eccentricity of

$$\epsilon_2 = \frac{h}{160}, \text{ or say } \frac{D}{160},$$

even a probability.

To sum up, in solid columns an eccentricity of loading

$$\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (420)$$

and in built-up columns an eccentricity of loading

$$\epsilon_2 = \frac{L}{1000} + \frac{h}{20} + \frac{h}{160} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (421)$$

should be expected and allowed for.

The author considers that no useful rules can be given for the probable eccentricities of cores in cast-iron columns. A rational design of cast-iron and timber columns is, however, hardly possible.

Turning now to the second method, namely that of determining the eccentricity from experimental data, two lines of attack may be followed. The observed deflection can be substituted in the expression for the deflection, and the eccentricity obtained therefrom; or a value may be chosen for the eccentricity, which substituted in the formula for the ultimate load will enable the average value of that load to be determined.

As has been pointed out (p. 130), the first of these leads to very variable results. Bauschinger's tables (1887) and Tetmajer's tables (1896) show this very clearly. The latter remarks that it is quite out of the question to suppose that the original eccentricity is a function of the radius of gyration, or that it remains constant during the experiment. The eccentricity is quite irregular and fortuitous, and it is not possible to explain the experimental results by assuming a fixed original eccentricity. From the tables quoted it appears that with his carefully mounted specimens $\epsilon_2$ varied from 0·003 to 0·708 cm. (the latter value occurred near the ultimate load), or if calculated while the loads were relatively small, $\epsilon_2$ varied from 0·003 to 0·102 cm. Bauschinger's figures fully confirm the conclusion that $\epsilon_2$ varies during the experiment. As to Tetmajer's first point that the eccentricity is not a function of the radius of gyration, it appears to be perfectly well founded. So far as can be seen, there

is absolutely no connexion between the two quantities, nor has anyone offered the slightest reason why one should be a function of the other. The only connexion proved between them is that with Neville's assumption (1902) regarding the variation in the modulus of elasticity

$$\epsilon_2 = \frac{\kappa^2}{D} \cdot \frac{E_1 - E_2}{E_a}.$$

The only apparent reason why $\epsilon_2$ is assumed to be a function of $\kappa$ is that it simplifies the formula. Even if it could be shown that the probable error in centering columns is proportional to their size, a very doubtful proposition, it would be more rational to assume that $\epsilon_2$ was proportional to D, and granting that size makes a difference, it is probable that length is a more important factor than width. It is, for example, probably easier to centre a specimen 12 in. across and 2 ft. long than one 8 in. across and 35 ft. long. Further, while it is easier to centre certain shapes than others, it is obvious that the radius of gyration is no measure of this. In short, the eccentricity is not a function of the radius of gyration. Unfortunately most writers have assumed that it is. The corresponding assumption that $\beta = \frac{\epsilon_2 v_2}{\kappa^2}$ is a constant is likewise utterly devoid of a rational basis, and only made to simplify the formula.

Morris (1911) used the deflection to determine the eccentricity from Buchanan's experiments, applying the method of least squares. He obtained the following values.

*Normal to the pins:*

| | Accidental Eccentricity | Probable Error |
|---|---|---|
| Maximum . . | + 0·728 in. | 0·027 in. |
| Mean . . . . | + 0·251 ,, | 0·024 ,, |
| Minimum . . | − 0·007 ,, | 0·022 ,, |

*Parallel to the pins* ($q = \frac{1}{2}$) *:*

| | | |
|---|---|---|
| Maximum . . | 2·539 in. | 0·093 in. |
| Mean . . . . | 1·175 ,, | 0·062 ,, |
| Minimum . . | 0·345 ,, | 0·074 ,, |

The value of $\epsilon_2$ varies all the way through the experiment, and is by no means constant for any one specimen.

For the reasons given on p. 130, however, the method is not a good one, and is bound to lead to variable results.

Although it is possible, as Ayrton and Perry (1886), Moncrieff (1901), and others have done, to give a reasonable interpretation of the deflection curve by the assumption of an original and constant eccentricity, the number of cases thus treated is much too small to furnish any general rule as to the magnitude of the eccentricity.

Dealing next with the interpretation of the ultimate strength of specimens by the assumption of an original and constant eccentricity, it will be found that here again it is possible to obtain reasonable results. Most writers have used Tetmajer's experimental results as a basis for their investigations.

Marston (1898) showed that the eccentricity formula would represent the average results of those experiments if the following constants were used :

| | *Wrought Iron.* | *Mild Steel.* |
|---|---|---|
| $\frac{\epsilon_2 v_2}{\kappa^2}$ | 0·07 | 0·06 |
| $f_e$ | 38,000 | 40,000 lb. sq. in. |
| E | 28,500,000 | 30,000,000 ,, ,, ,, |

Jensen (1908), using the same tests as a basis, found as an average value for the eccentricity

$$\epsilon_2 = 0\cdot036\ \kappa$$

and for $\beta$

$$\beta = \frac{\epsilon_2 v_2}{\kappa^2} = 0\cdot072.$$

The maximum value of $\epsilon_2$ was, however, $\epsilon_2 = 0\cdot1\ \kappa$, and Jensen suggests that for practical work a factor of safety should be introduced, making the eccentricity $\eta\epsilon_2$ where $\eta > 3$. Adopting the value $\eta = 5$, the practical value of the eccentricity becomes $\epsilon_2 = 0\cdot18\ \kappa$ and $\beta = 0\cdot36$. He remarks that when $\frac{L}{\kappa}$ was less than 100, $\epsilon_2$ was constant, when $\frac{L}{\kappa}$ was greater than 100, $\epsilon_2$ decreased markedly.

Ostenfeld (1898), applying the method of least squares to a variant of the eccentricity formula, found as a mean value for $\beta$ from Tetmajer's mild steel experiments

$$\beta = \frac{\epsilon_2}{\omega} = 0\cdot35$$

practically equal to Jensen's value for practical cases, and about five times the actual value as obtained by Marston and Jensen.

Prichard (1913) finds that the eccentricity formula will represent Lilly's experiments on mild-steel specimens with round ends $\left(\frac{L}{\kappa} > 40\right)$ if

$$\beta = 0\cdot06,\ \epsilon_2 = 0\cdot00375 \text{ in., and } f_e = 58{,}000 \text{ lb. sq. in.}$$

Moncrieff (1901), as the result of plotting a very large number of experiments, gives for the value of $\beta$ :

For the upper limit curve $\frac{\epsilon_2 v_2}{\kappa^2} = 0\cdot15.$

For the lower limit curve $\frac{\epsilon_2 v_2}{\kappa^2} = 0\cdot60.$

This latter value is used in the practical formula. It is ten times Marston's value for Tetmajer's experiments.

Basquin (1913) calculates the following values of $\beta$ from Morris's figures (see above).

| | | |
|---|---|---|
| *Perpendicular to the pins :* | Maximum | 0·20 |
| | Mean | 0·07 |
| | Minimum | 0·00 |
| *Parallel to the pins :* | Maximum | 0·52 |
| | Mean | 0·22 |
| | Minimum | 0·05 |

As a mean value based on various empirical formulæ he suggests

$$\beta = 0{\cdot}1 + 0{\cdot}001\frac{L}{\kappa}.$$

Pilgrim (1904) suggests as a value for the eccentricity

$$\epsilon_2 = i\mathrm{L}$$

where

$$\begin{aligned} i &= 0{\cdot}001 \text{ for wrought iron and mild steel.} \\ &= 0{\cdot}002 \text{ for cast iron.} \\ &= 0{\cdot}015 \text{ for timber.} \end{aligned}$$

These values were obtained from Hodgkinson's experiments.

Prichard (1908) suggests $\beta = \frac{1}{10} + \frac{1}{700} \cdot \frac{L}{\kappa}$ and Jonson (1908) estimates $\epsilon_2 = \frac{\kappa}{4}$. (See Worcester, 1908.)

Considère (1889) suggests $\frac{\epsilon_2}{\kappa} = 0{\cdot}05$ as a probable value, but makes an addition to this value to allow for lateral wind pressure.

Cain (1887) gives:

| | |
|---|---|
| Cast iron, position- and direction-fixed ends . . | $\beta = 0{\cdot}2$ |
| Wrought iron, „ „ „ „ „ . . | $\beta = 0{\cdot}2$ |
| Wrought iron, hinged ends . . . . . . . . | $\beta = 0{\cdot}1$ |

Kirsch (1905) considers that an inevitable eccentricity of 10 mm. should be assumed, and any direction-fixing neglected altogether. He would call the load which produces a deflection of $\frac{L}{100}$ the ultimate strength of the column, and use a factor of safety of two.

Müller-Breslau (1911) suggests an eccentricity of loading $\epsilon_2 = \frac{L}{200}$. Under these conditions the maximum stress $f_c = \frac{f_e}{2}$.

Kayser (1912) considers it sufficient to assume that $\frac{L}{75} > \epsilon_2 > \frac{L}{300}$, say as a mean value $\epsilon_2 = \frac{L}{150}$.

Jasinski (1894) finds as the worst likely value for the eccentricity $\epsilon_2 = 0{\cdot}05\ \kappa$ to $0{\cdot}1\ \kappa$, but would assume in addition a value $\epsilon_1 = 0{\cdot}001$ L for the initial curvature.

Findlay (1891) assumes that for hinged ends

$$\frac{\epsilon_1 + \epsilon_2}{v_2} = \text{constant} = c_1 = 0{\cdot}45,$$

$$\frac{1}{v_2}\left(\frac{\epsilon_1}{\pi^2} + \frac{\epsilon_2}{8}\right) = \text{constant} = c_2 = 0{\cdot}05,$$

and for position- and direction-fixed ends

$$\frac{1}{2} \cdot \frac{\epsilon_1}{v_2} = \text{constant} = c_1 = 0{\cdot}3,$$

$$\frac{1}{8\pi^2} \cdot \frac{\epsilon_1}{v_2} = \text{constant} = c_2 = 0{\cdot}0075$$

The maximum stress under these conditions is not to exceed $\frac{f_e}{2}$.

Alexander (1912) takes under normal conditions $\epsilon_2 = 0{\cdot}1\,\kappa$, which is to be increased, if the conditions are not very good, to $\epsilon_2 = \frac{1}{8}\kappa$.

Lastly, Krohn's artifice (1886), by which a direct estimation of the eccentricity is avoided, should be mentioned.

In the table following, these estimates are collected together with those calculated directly as previously explained.

INITIAL CURVATURE.—To the second great source of weakness in columns, initial curvature, the same amount of attention has not been paid as to eccentricity of loading. Nevertheless, Young (1807) coupled it with eccentricity of loading as one of the practical imperfections, and certain writers have claimed that it is the most important of the ills from which the column suffers (Hutt, 1912). In practical columns with direction-fixed ends it is a prime cause of weakness, for the eccentricity of loading merely increases the fixing moment.

There is no doubt that the perfectly straight member does not exist. Christie (1884), whose specimens were well straightened, says that although the bars were considered straight in a practical sense, refined measurement generally showed an appreciable curvature. Föppl (1897), who tested his specimens "as from the rolls," remarks that they had original deflections of from 1 to 3 mm. Their length varied from 2134·5 to 4133 mm. One was apparently perfectly straight, but its behaviour in the testing machine was very variable.

Where, in fact, an experimenter took the trouble to measure, he records as a rule considerable initial deflections. The Watertown Arsenal Reports are eloquent on this point.

The effects of initial curvature are much the same as those of eccentricity of loading, though it appears from Fig. 5 that with a given load and equal magnitudes of $\epsilon_1$ and $\epsilon_2$ the latter will produce the larger deflection. Several writers treat them as identical. Thus Ayrton and Perry (1886) propose for simplicity to assume initial curvature to be the sole imperfection, and to call the value of the equivalent initial deflection *

$$\epsilon = \frac{6}{5}\epsilon_2 + \epsilon_1.$$

Moncrieff (1901) and others propose to give a value to $\epsilon_2$ to represent both eccentricity of loading and initial curvature.

Many experimenters have noted the weakening effect of initial curvature; nevertheless, Lilly (1908), while recording that with long specimens some elastic deflection was always apparent before the ultimate strength was reached, says that he cannot agree with the view that the deflection was due to the

* The fraction $\frac{5}{6}$ given in their article is obviously a slip.

## ESTIMATED VALUES OF THE INITIAL DEFLECTION AND ECCENTRICITY OF LOADING

| Year. | Author. | To represent. | $\epsilon_1$ | $\epsilon_2$ | $\beta$ | Remarks. |
|---|---|---|---|---|---|---|
| 1822 | Tredgold | Practical conditions | — | $\frac{D}{2}$ | — | |
| 1865 | Francis | Practical conditions | $\frac{L}{300}$ | — | — | Cast-iron columns. Maximum value |
| 1886 | Ayrton and Perry | Hodgkinson's experiments on W.I. specimens | 0·0046 | — | 0·027 | Minimum values |
| | | | 2·59 | — | 30 | Maximum values |
| | | Practical conditions | $k_1 + k_2L^2$ | — | — | $k_1$ and $k_2$ are constants |
| 1886 | Fidler | Variation in the modulus of elasticity | — | $\kappa\frac{E_1 - E_2}{E_1 + E_2}$ | — | |
| | | | — | $\frac{1}{9}\kappa$ | — | Maximum value |
| 1887 | Cain | Experimental crippling loads | — | — | 0·2 | Cast-iron specimens, position- and direction-fixed ends |
| | | | — | — | 0·2 | Wrought-iron specimens, position- and direction-fixed ends |
| | | | — | — | 0·1 | Wrought-iron specimens, hinged ends |
| 1889 | Considère | Practical conditions | — | 0·05 $\kappa$ | — | To be increased for wind pressure |
| 1890 }<br>1896 } | Tetmajer | His experiments | — | 0·003 cm. | — | Minimum value |
| | | | — | 0·708 cm. | — | Maximum value |
| | | | — | 0·003 cm. | — | Min. } when load is relatively small |
| | | | — | 0·102 cm. | — | Max. } |
| 1894 | Jasinski | Practical conditions (probable values) | 0·001 L | 0·05 $\kappa$ to 0·1$\kappa$ | — | Both act together |
| 1896 | Pullen | Experimental results | — | 0·01 $\kappa$ | — | Upper limit |
| | | | — | 0·35 $\kappa$ | — | Lower limit |
| 1898 | Marston | Tetmajer's experiments | — | — | 0·07 | Wrought-iron specimens |
| | | | — | — | 0·06 | Mild-steel specimens |
| 1898 | Ostenfeld | Tetmajer's experiments | — | — | 0·35 | Mild-steel specimens |
| 1901 | Moncrieff | Experimental results | — | — | 0·15 | Upper limit |
| | | | — | — | 0·6 | Lower limit and practical formula |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1902 | Neville . . | Variation in the modulus of elasticity | — | $\frac{\kappa^2}{D} \cdot \frac{E_1 - E_2}{E_a}$ | — | |
| 1904 | Pilgrim . . | Hodgkinson's experiments | — | 0·001 L | — | Wrought-iron specimens |
| | | | — | 0·002 L | — | Cast-iron specimens |
| | | | — | 0·015 L | — | Timber specimens |
| 1905 | Kirsch. . . | Practical conditions | — | 10 mm. | — | Direction-fixing to be disregarded |
| 1908 | Jensen . . | Tetmajer's experiments | — | 0·036 $\kappa$ | 0·072 | |
| | | | — | 0·10 $\kappa$ | — | Maximum value |
| | | Practical conditions | — | 0·18 $\kappa$ | 0·36 | |
| 1908 | Prichard . . | Practical conditions | — | — | $\frac{1}{10} + \frac{L}{700\kappa}$ | |
| 1908 | Jonson . . | Practical conditions | — | 0·25 $\kappa$ | — | |
| 1911 | Müller-Breslau . | Practical conditions | — | $\frac{L}{200}$ | — | |
| 1911 | Morris . . | Buchanan's experiments | — | +0·728 in. | — | Max. normal to the pins |
| | | | — | +0·251 in. | — | Mean normal to the pins |
| | | | — | −0·007 in. | — | Min. normal to the pins |
| | | | — | 2·539 in. | — | Max. parallel to the pins ($q = \frac{1}{2}$) |
| | | | — | 1·175 in. | — | Mean parallel to the pins ($q = \frac{1}{2}$) |
| | | | — | 0·345 in. | — | Min. parallel to the pins ($q = \frac{1}{2}$) |
| 1912 | Hutt . . . | Practical conditions | $\frac{L}{500}$ | — | — | Sole imperfection |
| 1912 | Kayser . . | Practical conditions | — | $\frac{L}{75}$ to $\frac{L}{300}$ | — | |
| | | | | $\frac{L}{150}$ | — | Mean value |
| 1912 | Alexander . . | Practical conditions | — | 0·1 $\kappa$ | — | Normal conditions |
| | | | — | 0·125 $\kappa$ | — | Conditions not very good |
| 1913 | Prichard . . | Lilly's experiments | — | — | 0·06 | Mild-steel specimens, round ends |
| 1913 | Basquin . . | Practical conditions | — | — | $0·1+0·001\frac{L}{\kappa}$ | Based on various empirical formulæ |
| | | | $\frac{L}{960}$ | — | — | Maximum value which may pass inspection in the shops |

## ESTIMATED VALUES OF THE INITIAL DEFLECTION AND ECCENTRICITY OF LOADING (*continued*)

| Year. | Author. | To represent. | $\epsilon_1$ | $\epsilon_2$ | $\beta$ | Remarks. |
|---|---|---|---|---|---|---|
| — | Author . . | Practical conditions | — | $0{\cdot}001\,L$ | — | Eccentricity due to imperfect centering |
| | [see equations (420) and (421)] | Practical conditions | — | $\frac{1}{5}\frac{a_1\bar{v}_1}{a}$ | — | Eccentricity due to variations in E |
| | | Practical conditions | — | $\frac{h}{20}$ | — | Ditto in built-up columns |
| | | Practical conditions | — | $\frac{h}{160}$ | — | Eccentricity due to variations in area of flanges |
| | | Practical conditions | $\frac{L}{750}$ | — | — | Initial curvature of central axis |
| | (see Figs. 42 and 43) | Practical conditions | $0{\cdot}0023\frac{L}{\kappa}$ | — | — | Ditto (alternative estimate) |
| | | Practical conditions | $\frac{1}{10}\frac{a_1\bar{v}_1}{a}$ | — | — | Initial curvature due to variations in E (solid columns) * |
| | [see equations (422) and (423)] | Practical conditions | $\frac{h}{40}$ | — | — | Ditto in built-up columns * |

MORE RECENT ESTIMATES:—

| Year. | Author. | To represent. | $\epsilon_1$ | $\epsilon_2$ | $\beta$ | Remarks. |
|---|---|---|---|---|---|---|
| 1917 | Wylie. . . | Commercial mild-steel tubes | $\frac{L}{600}$ | $\frac{\text{int. diar.}}{40}$ | — | For aeroplane struts. (See Air Board Rules T.6, April 1917) |
| 1917–18 | Ross . . . | Experiment | — | $\frac{v_2}{20}+\frac{L}{600}$ | — | Equivalent eccentricity. (See R. W. Hawken, 1917–18) |
| 1919 | Robertson . . | Experiment and practical conditions | $\cdot003\frac{L}{\kappa}$ | — | — | Equivalent initial curvature (given in *Aeroplane Structures*, Pippard & Pritchard, London, 1919) |

The work of Smith (1887) and Findlay (1891) may also be consulted.

* If these values be assumed, the estimated values of $\epsilon_2$ due to variations in E are to be halved.

column not being straight, or to want of homogeneity of material. The same experimenter, however (1910), as the results of his tests on specimens with an intentional initial deflection, found that such an initial deflection had much the same effect as an equal eccentricity.

A number of writers have made an estimate of the probable magnitude of the initial curvature. Ayrton and Perry (1886) propose to assume that the equivalent initial deflection (see above) is

$$\epsilon = k_1 + k_2 L^2;$$

that is to say, they assume that the initial curvature of all specimens is constant, and that the error in centering is the same whatever the length. They determine $\epsilon$ from Hodgkinson's experiments on wrought-iron specimens, and find that

$$\epsilon \propto 0{\cdot}0046 \text{ to } 2{\cdot}59 \text{ in.}$$

$$\beta = \frac{\epsilon v_2}{\kappa^2} \propto \quad 0{\cdot}027 \text{ ,, } 30$$

The larger values of $\epsilon$ and $\beta$ are for the very long and thin specimens of Hodgkinson, and appear to be very doubtful.

Hutt (1912) would take $\epsilon_1$ to be the sole defect, and estimates its magnitude as

$$\epsilon_1 = \frac{L}{500}$$

Francis (1865) measured the deflection of ten hollow pillars $11\frac{1}{2}$ ft. long, 6 in. diameter, and $\frac{3}{4}$ in. thick, which supported a mill floor. None were quite straight. In a length of 10 ft. the loaded deflection averaged 0·03 ft., the maximum value being 0·08 ft. He suggests that the unloaded deflection would be

$$\epsilon_1 = \frac{L}{300}$$

Chew (1911) says that commercial columns, 30 ft. long, fabricated under the best conditions, seldom have less than $\frac{1}{4}$ inch of kink or bend in them. He proposes that the magnitude of the initial deflection which may pass inspection shall be standardized. Basquin (1913) endorses this proposal, and suggests $\frac{1}{16}$th in. per 5 ft. This is equivalent to

$$\epsilon_1 = \frac{L}{960}$$

Jasinski's estimate (1894)

$$\epsilon_1 = 0{\cdot}001 \text{ L}$$

has already been noticed. To this is to be added the eccentricity of loading.

It thus appears that some would make $\epsilon_1$ to be a function of L, some of $L^2$. It might also be argued that a bend or kink is more likely in a relatively thin than in a relatively stout specimen, in which case $\epsilon_1$ might reasonably be supposed to be a function of $\frac{L}{\kappa}$.

To obtain some exact information regarding the amount of curvature likely in practical members, the author has plotted two diagrams (Figs. 42 and 43).

In the first, the observed initial deflections recorded by various experimenters

have been plotted on a base line representing the length of the column in inches. In the second, the base line represents the value of $\frac{L}{\kappa}$ of the column irrespective of the end conditions.

In making the first of these diagrams, the initial deflection is plotted irrespective of its direction. The ordinate simply represents the maximum distance of any point on the central axis from the straight line joining the centres of area of the end cross sections.

In the second diagram the ordinate represents the initial deflection of the column in the direction corresponding to the value of $\frac{L}{\kappa}$ plotted, not the total

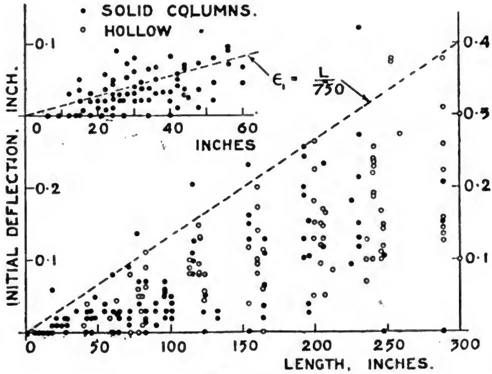



Fig. 42.

initial deflection as in the first diagram. In the vast majority of cases the plotted points have reference to the greatest value of $\frac{L}{\kappa}$ and least value of $\kappa$.

The observations used were not picked out in any particular way, but taken haphazard from those recorded by the author in his research. All are for wrought-iron or steel columns. The black circles represent solid members, either single rolled sections or built-up members of a solid character such as built-up beam sections. The hollow circles represent hollow sections, tubes, or lattice-braced columns. No "toy" specimens have been included, only such members as might be met with in practice.

It was observed that, in general, the original deflection of built-up solid members was on the whole smaller than that of simple sections, though some of the worst specimens were of the built-up type. The substratum of black circles on the $\frac{L}{\kappa}$ curve (Fig. 43) are chiefly Christie's (1884) specimens, which appear to have been exceptionally well straightened. They were, however, not very long, and appear on the length curve (Fig. 42) chiefly as a mass of

black circles near the origin. Many of the hollow circles were tubes tested at Watertown Arsenal.

The small upper diagram (Fig. 42) represents a number of experiments taken chiefly from Marshall's tests (1887) on unstraightened specimens "as from the rolls." Most of these specimens were of small cross section and could not be classed as practical members.

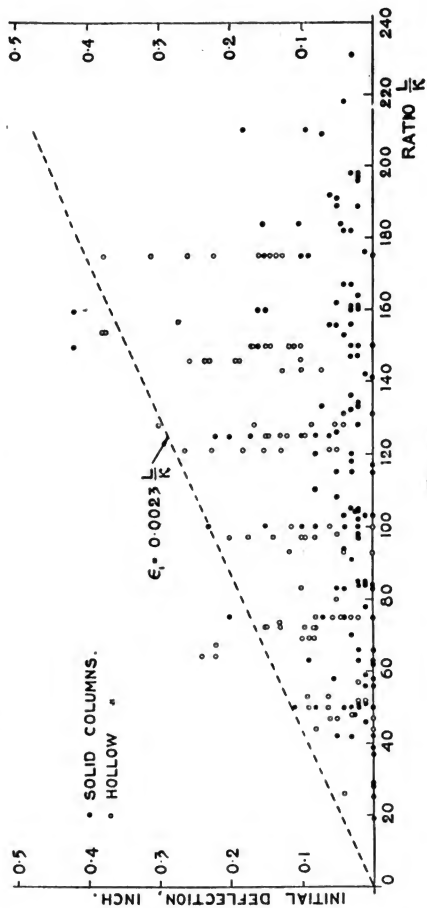



FIG. 43.

When it is reflected that these diagrams have reference for the most part to carefully prepared laboratory specimens, it will be apparent that no designer could be certain that a practical member, carefully straightened, would have an initial curvature of less than

$$\epsilon_1 = \frac{L}{750}$$

or

$$\epsilon_1 = 0{\cdot}0023 \frac{L}{\kappa} \text{ in.}$$

values represented by the straight lines on the figures; and that, if deductions may be drawn from Marshall's results, unless the members were so straightened, the initial deflection might be practically double this.

It does not appear from these figures that there is much to choose between the assumptions that $\epsilon_1$ is a function of L or of $\frac{L}{\kappa}$. Possibly on the whole the first assumption is the better. It does not seem probable that $\epsilon_1$ increases as $L^2$.

This value $\frac{L}{750}$ represents, then, the probable curvature of the central axis.

To this must be added the possible increase in the initial deflection due to variations in the modulus of elasticity. It has been shown that this variation may be equivalent to an eccentricity.

$$\frac{1}{5}\frac{a_1\bar{v}_1}{a} \text{ in a solid column}$$

and $$\frac{h}{20} \text{ in a built-up column.}$$

If, however, the modulus vary in the direction of the length of the column as well as in the width, part of this eccentricity may be in effect an initial deflection. Probably the most satisfactory assumption to make, when dealing with direction-fixed columns, is that one-half of the calculated value is an eccentricity, the other half an initial deflection. That is to say, the effect of variations in the modulus of elasticity in solid columns is an eccentricity of loading

$$\epsilon_2 = \frac{1}{10}\frac{a_1\bar{v}_1}{a},$$

plus an initial deflection $$\epsilon_1 = \frac{1}{10}\frac{a_1\bar{v}_1}{a};$$

or, in built-up columns, an eccentricity of loading

$$\epsilon_2 = \frac{h}{40} \text{ or say } \frac{D}{40},$$

plus an initial deflection $$\epsilon_1 = \frac{h}{40} \text{ or say } \frac{D}{40}.$$

To sum up, in position-fixed columns, in addition to the eccentricity of loading already calculated, an initial deflection of $\frac{L}{750}$ should be assumed. In position- and direction-fixed columns an initial curvature of

$$\epsilon_1 = \frac{L}{750} + \frac{1}{10}\frac{a_1\bar{v}_1}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (422)$$

in solid columns, and

$$\epsilon_1 = \frac{L}{750} + \frac{h}{40} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (423)$$

in built-up columns should be allowed for, and the estimated eccentricity due to variations in the modulus of elasticity taken at half the corresponding values for position-fixed columns.

As a first approximation, Hutt's estimate $\epsilon_1 = \frac{L}{500}$, or better, for the shorter columns, $= \epsilon_1 \frac{L}{400}$, may be used. In addition, the eccentricity of loading should be allowed for in position-fixed columns.

REDUCTION IN THE STRENGTH OF THE MATERIAL.—To the third great source of weakness in a column, deterioration in the quality of the material, not much consideration is paid in the usual formulæ. Nevertheless, more and more stress is laid by experimenters upon it.

Under this heading must be included:

- Effect of Past History.
- Effect of Cold Straightening.
- Stresses set up during Manufacture.
- Initial Stresses.
- Local Permanent Sets.
- Annealing.
- Flaws and Local Defects.
- Imperfect Castings.
- Knots and Shakes in Timber.

That the past history of the material has a great influence on the ultimate strength is well known. Baker (1888) remarked that nothing showed the influence of previous strains on steel better than experiments on long columns. The resistance to flexure of a solid mild-steel column thirty diameters in length varied according to previous treatment as follows:

| | Tons sq. in |
|---|---|
| Annealed | 14·5 |
| Previously stretched 10 per cent. | 12·6 |
| Previously compressed 8 per cent. | 22·1 |
| Previously compressed 9 per cent. | 28·9 |
| Straightened cold | 11·8 |

Considère (1889) made a special experiment on a bar of comparatively mild steel $\left(\frac{L}{\kappa} = 43{\cdot}25\right)$. After hot-rolling, the bar gave an ultimate resistance of 30 kg/mm². After testing it was cold rolled, which reduced its thickness by 10 per cent. Its ultimate resistance was then found to be 44·8 kg/mm², or greater than that of the much harder steels.

To observe the effects of cold straightening, Christie (1884) cold straightened a number of his specimens after the first test, and then retested them. When the bars were relatively long and the permanent bend slight, no diminution in strength was observed, but in the case of the shorter bars in which the distortion was more serious, the bars were about 10 per cent. weaker than before.

Howard repeatedly calls attention to the effect of internal strains and local permanent sets in promoting early failure in columns generally, and particularly in built-up members. He remarks (1908) that cold straightening in particular produces internal strains, causes early sets to appear, restricts the range of loads which may be applied before the elastic limit is reached, and consequently tends to lower the ultimate resistance.

Moncrieff (1901), Basquin (1913), and others have drawn attention to these effects.

The effect of annealing can be observed in the table above. Considère (1889) has pointed out how much the ultimate strength is reduced thereby. He found that whilst annealing reduced the ultimate tensile resistance by 4 per cent. only, it reduced the resistance to crippling by 9 per cent., and suggests that this is due to the elastic limit being reduced more than the ultimate tensile strength. He remarks that thick bars lose less by annealing

than thin ones, and that the best steels for columns are those which have been rolled at the lowest temperatures.

Lilly (1908), having tested his mild-steel specimens "as from the rolls," carefully straightened them, annealed, and then retested them. Their strength after annealing was found to be practically coincident with that of the wrought-iron specimens; which, he says, is sufficiently accounted for when the strengths of the material at the yield point and the respective elongations in the annealed and unannealed states are considered.

These phenomena are well explained on the assumption that, in ductile material, it is the elastic limit which is the most important factor in determining the ultimate strength of columns. To this conclusion, as will be seen, several experimenters have come on other grounds. Granting it to be the case, whatever raises the elastic limit increases the ultimate strength, whatever lowers the elastic limit diminishes the ultimate strength. Hence the importance of the past history of the material. Considère and Lilly have advanced this theory in explanation of the effect of annealing, Howard in explanation of the importance of local permanent sets. It will also explain the results given in Baker's table. The previous compressions mentioned would raise the elastic limit on the compression side, and hence the ultimate strength. The previous tension would raise the elastic limit on the tension side and correspondingly lower it on the compression side. Hence the ultimate strength would be lowered. Annealing would lower the elastic limit and hence the ultimate strength.

To estimate the magnitude of these effects and the consequent deterioration in the quality of the material is no easy matter. To a certain extent, of course, empirical column formulæ take account of such deterioration when their constants are determined from experimental results. The difficulty is that the past history of the test piece is as a rule quite different from that of the member. Considère suggested using test pieces which had undergone exactly the same processes as the member itself, but the design of the majority of columns has to be based on existing information.

Basquin (1913) and others have proposed to reduce the working stress by a certain amount to allow for deterioration and initial stresses.

This is much the same thing as increasing the factor of safety or adopting a lower working stress for compression than tension, expedients common in good engineering practice. Probably all that can be done in the present state of knowledge concerning the magnitude of the reduction in quality of the material is to adopt a working stress in compression at least 10, or better 20, per cent. lower than that in tension.

Regarding the effect of flaws and other defects in castings, these again must be allowed for by a suitable factor of safety, which should be all the greater in that a flaw not only reduces the available area, but throws the centre of resistance out of line.

The presence of knots and shakes in timber has a great influence in determining the strength of the material to resist compression. This will, however, be better discussed when the ultimate strength of timber specimens is being considered.

The publication (1917) of the Final Report of the Column Committee of the American Society of Civil Engineers has again drawn attention to the variation in strength caused by non-uniformity in the material, especially in heavy sections. For columns of steel with an ultimate tensile strength of 60,000 lb. sq. in., an elastic limit of 38,000 lb. sq. in., and an extension of

28 per cent., the Committee recommend a working load of 12,000 lb. sq. in. $\left(\frac{L}{\kappa} = 0 \text{ to } 80, \text{ flat ends}\right)$, corresponding to a factor of safety of about 2 on the elastic limit load of the columns; which for values of $\frac{L}{\kappa}$ varying from 0 to 155 failed at from 21,000 to 38,600 lb. sq. in.

**End Conditions.**—One of the most unsatisfactory features of the column problem is the question of end conditions. Here the practical designer is left with very scanty aid from either theory or experiment. His practical conditions are not only indeterminate, but differ entirely from both theoretical and practical end conditions, which in turn differ from one another. The indefinite nature of the practical conditions would be a serious addition to the difficulties of the column problem, were it not for the fact that most practical columns are relatively short.

THEORETICAL END CONDITIONS.—Euler, in his memoir, assumed that the column was perfectly fixed in position, and perfectly free in direction. He showed that under these conditions the shape of the elastic line was, for all practical purposes, part of a curve of sines. It was an easy deduction that under different circumstances a different part of the sine curve would represent the shape of the column. Lagrange investigated the question and determined the possible shapes if the ends be position-fixed. Others, of whom Duleau (1820) appears to have been the first, recognized the possibility that one or both ends might be held fixed in direction by external means, and so obtained a new series of shapes, which are now familiar as the "standard cases." Young (1807) had previously pointed out that an eccentrically loaded column might be looked upon as a portion of a concentrically loaded column of much greater length, and later writers have considered many different combinations of direction-fixing with eccentricity of loading.

The vast majority of the theoretical work on the subject has been, in fact, based on the assumption of perfect fixidity in position with either perfect freedom or perfect fixidity in direction.

EXPERIMENTAL END CONDITIONS.—Rather unfortunately, owing probably to its singular character, Euler's result so obsessed the minds of early experimenters that they endeavoured rather to verify or condemn the theory than to obtain experimental information as to the behaviour of columns under practical conditions. Modern experimenters have in nearly all cases followed the same course.

This tendency is quite evident in the early experiments of Duleau and others. Hodgkinson (1840), who was the first to experiment on any scale in a scientific way, modelled the whole of his work with the Eulerian theory as the underlying idea, even to his unhappy formula. He attempted to reproduce in the testing machine the "standard cases." His end conditions are shown in Fig. 44. His aim was freedom in direction or fixidity in direction; in short, the theoretical conditions. Similarly, the vast majority of modern experiments have been carried out with specimens mounted on points or knife edges, all designed to reproduce theoretical direction-freedom.

In America, it is true, flat or pin ends are usually employed, but to what extent even these represent practical conditions is open to grave question.

Now both theoretically perfect direction-freedom and theoretically perfect fixidity in direction are easy to conceive and to deal with mathematically. As will be seen, they are difficult to obtain in a concrete state.

The experimental end conditions may be divided roughly into three classes:

Round and pointed ends.
Hinged and spherical ends.
Flat and flanged ends.

Hodgkinson, to obtain direction-freedom, pointed the ends of his specimens only to find that the ends crushed under the load. He therefore adopted for the shorter specimens hemispheres, or, as he says, curves flatter than hemi-

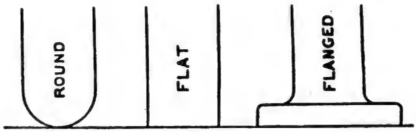



FIG. 44.

spheres. Even then he records (1857) that the ends flattened, and compressed circles about three-quarters of an inch in diameter showed not only on the specimens, but on the hard steel plates of the testing machine. The ultimate load was much increased by this. For perfect direction-freedom such ends evidently leave much to be desired.

Christie (1884), to improve matters, used a hard steel ball bearing on a hard steel plate. Even these were slightly flattened by repeated pressures, probably increasing the ultimate resistance. Not only so, but the use of carriers mounted on the ends of the specimens, which is the modern practice, instead of turning

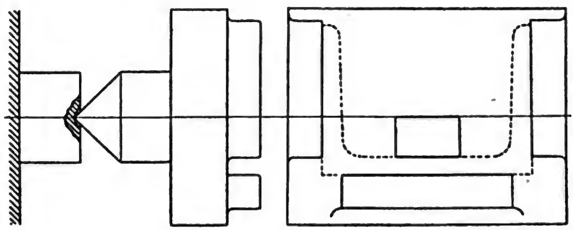


FIG. 45.

the ends of the specimen to the desired shape as Hodgkinson did, introduces another possible source of error, for it is difficult to ensure that the carrier is exactly centered.

In 1887 Bauschinger introduced his pointed ends (Fig. 45). Here the specimen was mounted on a carrier with a hard steel point instead of a ball, which point rested in a conical hole. With light loads it is possible that a comparatively perfect direction-freedom was obtained. As, however, the load increased, the point not only flattened, but bored into the hard steel seating, in some cases to the extent of as much as 7 mm., thus to some extent direction-fixing the ends. If the deflection curve of his specimen No. 2699*d* (1887) be

examined, it will be observed that the specimen began to deflect quite normally. When a load of 111 kg/cm² was reached, the rate of increase of the deflection suddenly became zero, and then again increased much more slowly. This may be due to the point penetrating into its seating, or perhaps to some grit getting in between the conical point and its seating (see some remarks of Föppl, 1897). It may even be due to imperfect contact between the specimen and the carriers. The effect is noticeable in several of the deflection diagrams. It is a singular fact that the ultimate strength of his specimens exceeded the Eulerian crippling load in so many cases. The difficulty in centering the carriers noticed above applies equally, of course, to those with pointed ends.

Tetmajer (1890) improved on Bauschinger's points by increasing the angle of the cone to 114°, and rounding the point more. Nevertheless, the difficulties were not entirely overcome, for he reports that little by little the point impressed itself into the seating, and it became necessary to adjust them afresh. Gérard (1902) remarks that in thirty-three cases out of 103 in Tetmajer's experiments the crippling load exceeded Euler's value, in some cases by as much as 22 per cent. Moreover, as Kármán (1910) has pointed out, the eccentricity of loading in Tetmajer's experiments was not inconsiderable, and this is confirmed by the latter's own calculations. The shape of Bauschinger's deflection curves suggests the same thing. It is proper to remark, however, that part at least of this apparent eccentricity may be due to initial curvature of the specimens. Föppl (1897), who actually measured the angular movement of his pointed ends, concluded that although the freedom in direction was almost perfect, yet small fixing moments existed due to the points flattening, or to the presence of grit in the clearance spaces.

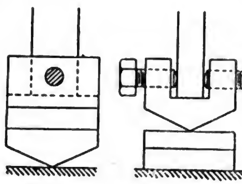


FIG. 46.

It may be concluded, therefore, that specimens mounted on carriers with pointed ends are not perfectly direction-free, nor is it possible perfectly to centre the load.

Considère (1889, 1894), who fully recognized these objections to the use of pointed ends, proposed to overcome the imperfections by the use of a system of double knife edges (Fig. 46), on which the specimen might be adjusted in position while in the machine. The knife edges give almost perfect direction-freedom in both directions, they will carry considerable loads without flattening or indenting the bearing plates, and the correct position of the specimen relative to the carriers can be determined experimentally by trial and error.

Specimens so mounted undoubtedly form the nearest approach to the theoretical conception of perfect direction-freedom, and the remotest divergence from practical conditions that can well be imagined.

The system incidentally has two minor disadvantages. First, the length of the specimen is different in the two planes of bending, and therefore unknown when the specimen deflects in a plane making an angle with the length of the knife edges. Secondly, as Kármán has pointed out, in the case of short specimens the length of the carrier forms an appreciable fraction of the length of the specimen, and a correction to the apparent length is necessary to allow for this. This second disadvantage is, of course, common to all specimens tested on carriers.

Single knife edges have been used with success by several experimenters.

Kármán in 1910 adopted this form for his rectangular specimens, and Lilly (1910) mounted his cylindrical specimens on carriers rotating about knife edges. Grooves on the carriers determined the amount of the intentional initial eccentricity, if any. This experimenter found that with long specimens, when the load producing failure was small, the same failure load was obtained whether the specimens had round ends or were mounted on knife edges. When $\frac{L}{\kappa}$ varied from 40 to 120 the knife edges gave more accurate results, for the round ends were more or less flattened. Nevertheless, care was necessary to centre exactly the specimens. When $\frac{L}{\kappa}$ was less than 40 either knife edges or round ends were used without much variation in the loads producing failure.

It may be concluded that with a system of knife edges the direction-freedom is as nearly perfect as is possible under ordinary experimental conditions, but that with the other varieties of end conditions adopted, the direction-freedom only approaches perfection under the lightest loads, and gets steadily worse as the load increases.

The second class of experimental end conditions includes the hinged and spherical types commonly used in America. The ends of the specimen are either like the pin connexions found in bridges or else a ball carrier is used working in a socket. There are essential differences in the two types, but they have many features in common.

It is customary to argue that hinged bearings form the nearest approach to practical conditions of any of the experimental end conditions, since they are of the actual type used in bridges. Granting this to be the case, nothing could indicate better than their behaviour the uncertainty of practical end conditions, and incidentally of experimental end conditions, when such types are employed. With no other form of end bearing will such wide ranges in the ultimate strength of the specimens be found. This uncertainty regarding the behaviour of such bearings has been noted by all experimenters. The fit of the bearings, the condition of the bearing surfaces, and the relative diameter of the pin or ball to the size of the specimen, all have considerable influence in determining the ultimate strength of the specimen, and on its experimental history. Christie (1884) remarks that the diameter of the ball or pin exercised a marked influence on the resistance of the bar, as also did the fit of the pin. If the bar be straight and accurately centered on the ball or pin, and if the latter be of substantial diameter and well fitted, the hinge-ended specimens will be fully as strong as the flat-ended. In fact, the resistance of the best hinge-ended specimens exceeded that of the best flat-ended specimens. On the other hand, the lowest of the hinge-ended approximated very closely to the average of the round-ended specimens. The special experiments of Christie are worthy of attention in this connexion. Speaking of Marshall's experiments (1887), which had relatively large pins, Considère (1894) remarks that they vary so much between themselves that they form their own condemnation.

Regarding the influence of the diameter of the pins, some experiments made at Watertown Arsenal (1883-4) are instructive—the effect of increasing diameter is plainly evident. In this connexion Cooper's remark should be borne in mind, that it is the *relative* size of the pin which is important.

If the pin be relatively large, a close fit, and not lubricated, there is no doubt that it may be quite as efficient a device for direction-fixing the ends as a flat plane; in fact, as will be seen, it may be even better. On the other hand,

with badly fitted, relatively small pins there may be considerable freedom in direction. Hence arises the large variation in the results obtained. Christie says that under ordinary circumstances hinge-ended specimens rotate on their hinge ends from the start. When correctly centered no such rotation occurs at the beginning of the deflection, but the bar bends like a flat-ended specimen until the point of failure is reached, when it rotates on its ends suddenly and with so much force that it may even spring from the machine. This remark of Christie's calls attention to an experimental phenomenon peculiar to hinged or spherical-ended columns. At a certain load and deflection, not of necessity the maximum load or ultimate strength of the column, the bending moment at the ends grows so large that it overcomes the frictional moment of the pins, causing them to rotate suddenly. The curvature of the specimen changes practically instantaneously from a reversed curve to a simple arc, the deflection increasing greatly. This phenomenon appears to occur in all specimens above a certain ratio of $\frac{L}{\kappa}$, which suggests that a certain critical deflection, or combination of load and deflection, is necessary.

Thus in the Watertown Arsenal experiments (1883–4), on square bars with pin ends, all the specimens for which $\frac{L}{\kappa} > 98$, except two, deflected suddenly after the maximum load had been passed, the resistance of the bar dropping in some cases 50 per cent. In the 1909–10 experiments on lap-welded tubes with spherical ends, when the value of $\frac{L}{\kappa}$ was 94 or greater, the deflection, after the maximum load had been passed, continued to increase under a reduced load up to a certain point, when it suddenly increased greatly. A similar phenomenon was observed in the case of the lap-welded tubes with pin ends when $\frac{L}{\kappa}$ was not less than 97. Again, in the case of the rolled steel beam sections when $\frac{L}{\kappa}$ was not less than 100, the specimens having passed the maximum load suddenly deflected greatly. The built-up beam sections in which $\frac{L}{\kappa}$ was not less than 100 deflected suddenly when the maximum load had been reached. Similarly, in the 1910–11 Report many cases were observed in which the specimens deflected suddenly when the maximum load had been reached or had been passed. In this series the phenomenon was observed with values of $\frac{L}{\kappa}$ as low as 76 (pin ends).

It should be observed here that in none of these experiments was any attempt made to determine a limit for $\frac{L}{\kappa}$ below which the column did not spring suddenly, but in each series the value of $\frac{L}{\kappa}$ in the specimens tested increased regularly up to about $\frac{L}{\kappa} = 150$ (see Howard, 1908).

The phenomenon, it is evident, may occur at the point of maximum load or considerably after it. It should probably be looked on rather as a consequence of the failure of the specimen than as its cause.

In some specimens the rotation is gradual and not sudden, the bending moment overcoming the frictional moment, the pin rotating, coming to rest, and then slipping again, and so on. This may be seen in Exp. No. 155 of Christie's experiments (1884).

Basquin (1913) has pointed out that just as the frictional moment is capable of resisting rotation and so tending to direction-fix the ends of the column, it may be the cause of an initial eccentricity.

Several writers have given a mathematical analysis for the behaviour of pin-ended columns. Findlay (1891) and Basquin (1913) may be consulted.

It is not difficult to allow for the effect of a frictional moment at the ends of a column. It is exceedingly difficult to determine the magnitude of that moment in practice, depending as it does on the fit of the pin and the condition of the surfaces in contact. This has led many to the safe conclusion that pin ends cannot be regarded, for practical purposes, as better than position-fixed ends. In the majority of empirical formulæ, however, they are placed somewhere between position-fixed and flat ends.

The third class of experimental end conditions includes flat, flanged, and other end conditions designed to produce fixidity in direction.

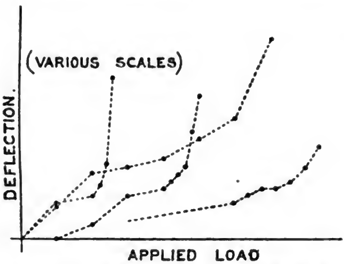



Fig. 47.—Deflection Curves for Flat-ended Specimens (various Experimenters).

Many experiments have been made on specimens with flat ends, and not a few writers have argued that such experiments approach more closely to practical conditions than any others.

It is usually supposed that flat-ended specimens behave as direction-fixed columns up to some point when they "swing round," that is to say, the ends cease to bear fully on the crossheads of the testing machine and rotate about one edge of the cross section. The specimen then becomes a position-fixed column. The load at which "swinging round" occurs is usually looked on as the ultimate strength of the column, and various formulæ have been suggested to determine it. This theory appears to have been given first by Lamarle (1846).

A careful consideration of the tests on flat-ended columns leads, however, to a very different conception of their behaviour. If the deflection curves of these members be examined, it will be found that instead of a continuous curve as is obtained with position-fixed columns, the curve, as shown in Fig. 47, in nearly all cases begins like that in a position-fixed column, and at some spot stops and starts again. This point is evidently not that at which "swinging round" occurs, for the rate of increase of the deflection, instead of becoming more rapid, actually becomes zero, and then increases slowly. Not only so, but the initial rate of increase of deflection is always much greater than that after the change. This behaviour is characteristic, and can be observed in nearly all flat-ended specimens. There is little doubt that it is occasioned by bad contact between the ends of the specimen and the crossheads of the testing machine. At the beginning of the experiment a state of affairs similar to that shown in Fig. 24 or Fig. 26 exists, and instead of behaving as a direction-fixed column, the specimen deflects as a very greatly eccentrically loaded position-

fixed column, until first one end and then the other come to a good bearing, after which the column begins to deflect as a position- and direction-fixed member. Basquin (1913), as a result of his examination of the Watertown Arsenal experiments of 1883, comes to a similar conclusion. The kink or kinks in the deflection curve indicate the change from one state to another. There are even cases on record (Christie, 1884, Exp. No. 6) where it appears probable that the ends of the column were slightly convex, and it continued to deflect as a position-fixed specimen. The ends of No. 33 (Christie, 1884) probably never bore fully on the crosshead at all.

The evidence in favour of the above theory appears to be conclusive. So careful an experimenter as Bauschinger (1887) remarks that great trouble was experienced in getting the load uniformly spread over the flat ends of his specimens. Often a satisfactory result could only be attained by grinding down the ends of the specimen by repeatedly moving it to and fro on the crosshead plates, and repeatedly straightening the latter.* Any deviation from uniformity had a greater influence on the modulus of elasticity and the elastic limit than on the yield point and ultimate strength. With regard to the latter remark, it is evident that untrue bearings would affect the earlier stages of the experiment more than the later ones.

Bauschinger's remarks (1887) on the variable nature and direction of the deflection in his flat-ended specimens have a distinct bearing in this connexion, as has also the fact that he and others have recorded S-shaped bending in experiments on flat ends.

Christie (1884) remarks that irregular deflection was more frequent in flat-ended than in hinge-ended specimens. He records that in

13 flat-ended,
4 hinge-ended,
0 round-ended,
0 flange-ended

specimens the deflection decreased as the load increased, due to the direction of greatest deflection being reversed under the strain.

These effects are at once explained by a theory of imperfect end bearings, and a glance at Fig. 24 will at once show why the direction of the deflection may reverse with increasing loads.

Many experimenters have noted that the end bearings were bad, and some have had to pack them up. Thus, in the Watertown Arsenal experiments of 1879–81, it is reported that brass strips had to be inserted to bring the ends to a good bearing. Tetmajer and Bauschinger left the spherical bearings of the crossheads of their testing machines loose, and only tightened them up after the initial loads had been applied.

It must not be assumed that imperfect bearing is the effect merely of careless experimenting. That the experiments of such men as Hodgkinson, Bauschinger, and Howard of Watertown Arsenal, exhibit these characteristics is sufficient evidence that the practical difficulties are such that perfect contact can never be relied on. Neither must it be imagined that the first stage of the experiment, before the ends begin to bear fully on the crosshead, is a very small and unimportant portion of the total life of the specimen. From Fig. 47 it will be observed that the applied load may even exceed one-half of the ultimate load before the specimen becomes direction-fixed, and, as has been seen, it is

* See further the remarks in the Progress Report of the Special Committees on Steel Columns. Amer. Soc. C.E. (1914).

probable that in some cases the specimen never becomes direction-fixed at all. These experiments were made, be it observed, by careful experimenters. In practice the working load would be limited to one-fifth or one-quarter of the ultimate, so that if such columns could exist in practice they would behave during their whole existence as eccentrically loaded position-fixed columns, and not as direction-fixed columns at all. In fact, Tredgold's suggestion (1822) that the load line in all flat-ended columns should be taken as passing through the extreme edges of the cross section has some justification.

The experimental life of a flat-ended specimen must therefore be divided into three stages.

1st. From the initial application of the load until the ends bear all over it will deflect as an eccentrically loaded position-fixed specimen.

2nd. From the moment of full contact until the maximum load is reached it will behave as a position- and direction-fixed column with an initial curvature.*

3rd. After the maximum load has been passed, it will continue to deflect, possibly under a reduced load, until when the deflection attains a certain value

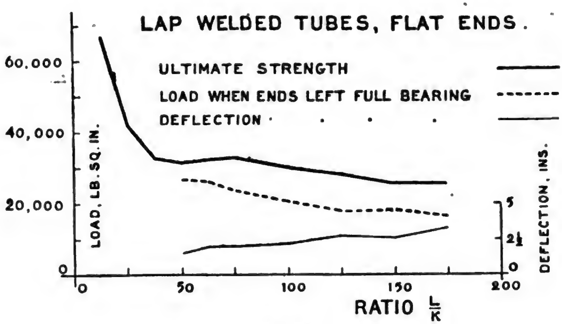



Fig. 48.

it will "swing round" one edge of the end cross section, which will leave the crosshead, and if the experiment be continued, the specimen will go on deflecting under a greatly reduced load as a position-fixed column. "Swinging round" may and does occur in certain cases under the maximum load, but does not of necessity occur when that load is reached. It may be well to review the experimental evidence regarding this point. Christie (1884) records that short flat-ended specimens remained solidly seated. The longest always rotated on their ends, and sometimes showed a tendency to do so before the ultimate resistance was attained. In the 1908–9 experiments made on lap-welded steel tubing at Watertown Arsenal, "swinging round" occurred for all ratios of $\frac{L}{\kappa}$ from 25 to 175, and in all cases but one the maximum load was passed before the ends rotated, and rotation took place under much reduced loads (see Fig. 48). Similar phenomena were observed in the experiments on the rolled beam sections.

* During the first part of stage two the direction-fixing moment will be negative (see some remarks on p. 79).

In the 1909–10 experiments on built-up beam sections, when the value of $\frac{L}{\kappa}$ was not less than 125, the specimen having reached the maximum load suddenly sprung sideways. In the 1910–11 experiments on similar sections "swinging round" occurred when $\frac{L}{\kappa} = 100$.

It does not appear from the above, therefore, that the condition for "swinging round" determines of necessity the maximum load which the column will withstand. Nor is there a definite value for $\frac{L}{\kappa}$ above which "swinging round" must occur and below which the specimen will remain solidly seated. From equation (207) it would appear that the condition for "swinging round" is complicated, and certainly not a simple function of $\frac{L}{\kappa}$. The phenomenon, like that observed in pin-ended specimens, appears to be the result rather than the cause of failure. What appears to happen is that the deflection under the maximum load increases so rapidly that "swinging round" occurs before the load can be removed. It is clear that there is a critical value both for the load and the deflection, and hence the phenomenon is more likely as $\frac{L}{\kappa}$ increases.

Lamarle (1846) found as the critical value for $y_0$ under ideal conditions:

$$y_0 \geqq \frac{D}{8} \text{ for a circular cross section,}$$

$$y_0 \geqq \frac{D}{6} \text{ for a rectangular cross section,}$$

and showed that "swinging round" would occur for values of $\frac{L}{D}$ as low as 2 provided that the crippling load was exceeded by 1 per cent. only. These figures were based on the Eulerian theory, and probably are not of much practical value, though Tetmajer (1890, Table No. 6) records that specimens in which $\frac{L}{D} = 2{\cdot}9$ rotated on one edge.

Pearson (1886) gives as the critical value for the deflection

$$y_0 = D,$$

which appears to be much too large.

Against the theory that "swinging round" occurs after the maximum load has been attained, due to the deflection reaching a critical value, must be put the evidence that in some columns at least "swinging round" is incipient before the ultimate load is reached, and may therefore be a factor in determining the same. A remark of Christie's to the effect that some specimens showed a tendency to turn before the ultimate resistance was attained has already been quoted. Bauschinger (1887), having remarked that after the ultimate resistance had been exceeded the end sections rotated on one edge, and the deflection curve of the column became a single arc, goes on to say that preparation for "swinging round" takes place long before the ultimate

resistance is reached, for the Tables show that the new curve combines with the old one, and the tangents to the ends of the central axis become inclined to the normal.

In explanation of this, it is possible that in long columns the critical deflection is reached before the critical load. "Swinging round" may then be looked on as incipient, and the condition of affairs spoken of by Bauschinger would exist. When the load attains the critical value, the ends would at once leave the crosshead. In short columns, however, the critical load would be attained first and the column would remain firmly seated until the deflection reached its critical value, when it would "swing round."

The discussion whether or not "swinging round" is the determining cause of failure or only an accompanying phenomenon is, however, really only of academic interest, for no practical column ever behaves as does a flat-ended one in the testing machine. The slightest rigid connexion with another member, inevitable in practice, partially direction-fixes the ends, and the whole condition of affairs is altered.

In the same class with flat ends must be included those end conditions designed to produce fixidity in direction. To direction-fix the ends of his specimens, Hodgkinson had large flanges cast on their ends (Fig. 44). Other experimenters have clamped flanges on the ends of their specimens. In some Watertown Arsenal experiments (1909–10) the specimens were clamped to the crossheads of the testing machine.

None of these methods produces true direction-fixing. With the flanges the difficulty is to get them true and parallel, with the result that the same phenomena are met with as in the case of the flat-ended specimens, and the kink in the deflection curve due to the imperfect bearings may often be observed.

If the specimens be clamped to the crossheads, it is almost inevitable that bending moments will be set up at the ends, which not only destroy the perfection aimed at, but in addition are totally unknown in magnitude.

Further, it is difficult to believe that in any case the crossheads of the testing machine possess the necessary amount of rigidity to prevent absolutely any angular movement, particularly if the specimens are of any size.

Few realize how very small the inclination to the perpendicular of the average column is, even under the ultimate load, particularly if the column be short. From a few examples picked at random from the 1909–10 Watertown Arsenal experiments on welded tubes the following figures are obtained :—

| Exp. No. | End Conditions. | $\frac{L}{\kappa}$ | Ultimate Strength. | Applied Load. | Length. | | Deflection. $\Delta$ | $\frac{\Delta}{L}$ |
|---|---|---|---|---|---|---|---|---|
| | | | lb. sq. in. | lb. sq. in. | ft. | in. | in. | |
| 1927 | Spherical . . | 47 | 33,650 | 33,000 | 6 | 5½ | ·042 | ·00054 |
| 1915 | ,, . . | 150 | 26,760 | 26,000 | 20 | 6¾ | ·21 | ·00085 |
| 1947 | Pin . . . | 47 | 33,470 | 33,470 | 6 | 5½ | ·067 | ·00087 |
| 1932 | ,, . . . | 175 | 19,370 | 15,000 | 24 | 0 | ·156 | ·00054 |
| 1974 | Direction-fixed . | 44 | 32,000 | 32,000 | 5 | 11⅞ | ·315 | ·0044 |
| 1958 | ,, ,, . | 146 | 26,910 | 26,000 | 20 | 0·85 | ·377 | ·0016 |

An even better conception of the smallness of the deflection may be obtained from Fig. 70, where the actual movements of the centre points of Nos. 1915 and 1947 are plotted to a scale of *five times* the actual size.

The deformation so apparent in published photographs takes place *after* the maximum load has been passed, under very likely a very much reduced load.

Föppl (1900) tested a cast-iron flanged column, and to ensure that the load was uniformly spread over its base, he inserted a packing of felt and cardboard. The result was that the column behaved as a direction-free specimen.

It might be possible, by the introduction of adjustable bending moments at the end of a specimen, to produce true direction-fixing experimentally; but no one appears to have made the endeavour.

The few experiments made on direction-fixed specimens have, in addition, been made for the most part on specimens in which the value of $\frac{L}{\kappa}$ was very large, much larger than would be met with in practice; and no attempts have been made to ascertain the efficiency of the direction-fixing. In some experiments it was evidently poor.

This paucity of experimental data is a pity, for nearly all practical columns have direction-fixed ends, although it is true that the direction-fixing is in most cases imperfect.

Before passing to the consideration of practical conditions, it may be well to make a comparison of the different classes of end conditions. In the first place it is necessary to rid one's mind of the conventional ideas based on the Eulerian theory. The Eulerian ratios between the crippling loads for the different end conditions have no validity in practice. For columns with position- and direction-fixed ends Euler's formula loses its validity at a value of $\frac{L}{\kappa} = 200$ approximately, a length ratio much beyond that usual or proper in practice. Hence it follows that the Eulerian ratios between the crippling loads lose their validity at $\frac{L}{\kappa} = 200$ too, and that the true ratios between the ultimate strengths of practical columns are not constant, but vary with the ratio $\frac{L}{\kappa}$. This conclusion is amply confirmed by experiment.

Hodgkinson (1840) found that the mean ratio of the breaking loads of round-ended to those of flat-ended columns was 1 : 3·167. When $\frac{L}{D}$ was less than 30 the ratio decreased as $\frac{L}{D}$ decreased. Its maximum value was 1 : 2·360 when $\frac{L}{D} = 26$, and its minimum value was 1 : 1·395 when $\frac{L}{D} = 10$. The ultimate strength of a long flanged specimen was, however, equal to that of a round-ended specimen of the same diameter and one-half the length. When one end of a pillar was flat and the other round, the strength was always an arithmetical mean between the strengths of pillars of the same dimensions but with both ends flat and both ends rounded.

The results of Christie's experiments (1884) show clearly the variation in the ratios of the crippling loads with the value of $\frac{L}{\kappa}$ (see Fig. 49). He remarks that flange-ended members gradually gain in relative strength from short

lengths upward until, when $\frac{L}{\kappa}$ becomes about 500, they will be about twice as strong as either flat-ended or hinge-ended members. Round-ended members continually lose in relative strength, until when $\frac{L}{\kappa}$ is about 160 they will be about one-half as strong as flat-ended members, and when $\frac{L}{\kappa}$ is about 450 they will be about one-half as strong as hinge-ended members. When $\frac{L}{\kappa}$ is less than 20 there is no practical difference between the strength of the four classes, so long

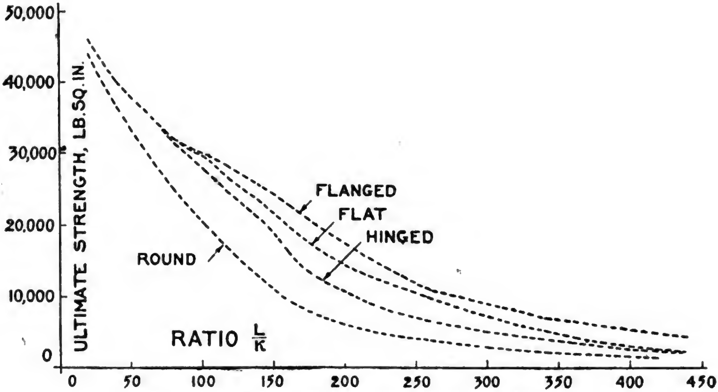



FIG. 49.—Average Strength Curves (Christie, 1884).

as reasonable care be taken to keep the centre of pressure in the centre of the specimen.

Burr, in 1884, commenting on Christie's experiments, says that below $\frac{L}{\kappa} = 120$ the end conditions are of little or no consequence, for the resistance is essentially the same whether the ends be hinged or fixed.

Kirsch (1905), whose direction-fixing appears to have been by no means good, gives the following ratios :—

RATIOS OF CRIPPLING LOADS

| End conditions. | Euler. | Experiment. $\frac{L}{\kappa} = 200$ | Experiment. $\frac{L}{\kappa} = 100$ |
|---|---|---|---|
| Both pointed . . . . | 1 | 1 | 1 |
| One pointed, one clamped | 2 | 1·78 | 1·05 |
| Both clamped . . . . | 4 | 2·99 | 1·13 |

It would appear, in fact, that for small ratios of $\frac{L}{\kappa}$ there is no advantage in direction-fixing the ends of the specimen provided that the load be concentric, and it is possible that short columns with hinged ends may be stronger than short columns with flat ends (see Fig. 50), due to the pin-ended specimens being better centered, and the trouble of imperfect bearings in the flat-ended columns.

Commenting on the 1908 and 1909 experiments at Watertown Arsenal, Howard (1909) remarks that they confirm the fact, as it is believed to be, that with axial loads the ultimate strength of all well-made compression members is the same for all types of end bearings (see, for example, Figs. 49 and 50).

This conclusion is of great practical importance in view of the indefinite nature of practical end conditions and the want of evidence concerning them. It appears that below $\frac{L}{\kappa} = 100$ there is practically no difference between pin,

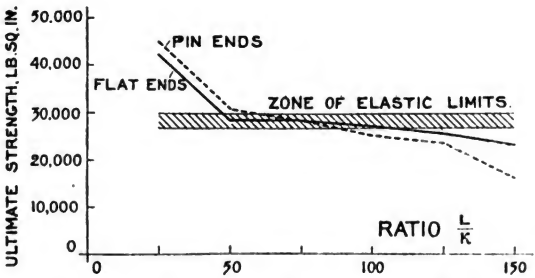



FIG. 50.—Tests on Rolled I-Beams at Watertown Arsenal (Howard, 1909).

flat, or flanged ends; that the curve for Christie's round ends falls below the other types is not improbably due to the effect of eccentricity of loading. However that may be, the ultimate strength in all the types of end conditions in which direction-fixing moments are set up is practically equal, and it would appear, therefore, not unjustifiable to conclude that the ultimate strength of a column with practical end conditions (imperfectly direction-fixed ends) would not differ much from the other types.

PRACTICAL END CONDITIONS.—From the foregoing it will be clear that, with the possible exception of specimens mounted on knife edges, the end conditions usual in experiments differ entirely from those assumed in theory, in spite of the fact that deliberate efforts have been made to make them agree. Not only so, but the end conditions which obtain in practice differ entirely from both the theoretical and the experimental end conditions.

No column in practice is either perfectly direction-free or perfectly direction-fixed. Nobody ever stood a practical column on knife edges or left its flat ends without some definite connexion to some other part of the structure. It is true that experiments on specimens with pin ends may conform more or

less to the practical conditions of columns with pin ends, but otherwise experimental and practical conditions are quite different.

De Préaudeau (1894) attempted in his experiments to reproduce the condition of bridge compression members by riveting his specimens to short lengths of bridge flange. But he mounted the whole thing on knife edges, so that the flange merely became part of the column, which vitiated entirely this attempt to reproduce practical conditions in the testing machine. Otherwise, as far as can be ascertained, no attempt has been made to experiment under practical conditions.

In practice all ends are *imperfectly direction-fixed.* The column is always more or less rigidly connected to adjacent members. It cannot deform without deforming them, and in turn it may be deformed as a consequence of their deformation. Even if the ends be flat or flanged, there is always some more or less rigid connexion which comes into play the moment deformation begins. No state corresponding to the initial state of the flat-ended member in the testing machine can or should exist. *It is almost impossible to imagine a practical end connexion which is incapable of transmitting a bending moment.*

(The only exception * to these remarks which occurs to the author is the upper end of a column carrying a detached load. Even in this case they apply to the lower end. They are, in any event, true of the vast majority of practical columns.)

Nevertheless, neither the connexion nor the adjacent member will, in general, be sufficiently rigid perfectly to direction-fix the ends of the column. Hence it follows that in practice all ends are imperfectly direction-fixed.

Strangely enough, in view of this fairly obvious conclusion, very little attention, either theoretical or experimental, has been paid to such ends. Dupuy (1897), finding that the direction-fixing of the ends of his specimens was not perfect, gave an analysis based on the Eulerian theory for imperfect direction-fixing, by which he interpreted his experiments. Murray (1913) assumed the deflection curve of the column to be "composed of a curve of versed sines, with a superimposed curve of sines." Certain German writers, including Winkler, Manderla, and Müller-Breslau, have considered the more general question of the reaction moments at the panel points of braced structures. Otherwise, with the exception of the analysis given in Part II of the present work, little appears to have been done on the subject. The difficulty is that no data exists on which theoretical work may be based.

The problem is not easy. The end conditions depend not only on the stiffness of the column itself, but on the rigidity of the adjacent members, and still more on that of the connexion between them. It is evident that the imperfectly direction-fixed column may vary from an almost position-fixed column on the one hand to a theoretically perfect direction-fixed column on the other. No simple solution is possible, but some direct experimental evidence of whereabouts between these two limits the practical column lies, under normal circumstances, could at least be obtained. What is required is an answer to the question: Through what angle will the end of the column turn when the load comes on it? Some observations on actual practical columns or on experimental columns under practical conditions, similar to those made by Föppl (1897) on specimens with pointed ends, would determine the probable magnitude of that angle, from which the probable value of the coefficient $k$ could be obtained, when the formulæ of Part II, Variation 7, might be applied to design the column.

* Aeroplane columns should perhaps be added.

In the meantime, the significance of the fact that, owing to the relative shortness of the practical column, an exact knowledge of the end conditions is not of extreme importance, should not be overlooked. As a result of the comparison of the experimental end conditions, the conclusion was reached that provided that $\frac{L}{\kappa}$ be less than 100, the ultimate strength of columns with any sort of direction-fixed ends is about the same. Hence it follows that the ultimate strength of such pin-, flat-, or flange-ended columns may be taken as representing approximately the ultimate strength of practical columns with direction-fixed ends. The working load may therefore be determined by the use of an appropriate factor of safety, or so it would appear.

It is important to distinguish, however, between the conditions under the working and ultimate loads. Even though the ultimate strengths of the various types be about equal, there is no doubt that when the load is equal to about one-quarter of the ultimate strength, that is to say, under the working loads, the condition of affairs in the different types is utterly dissimilar, and in none of the experimental types does it approximate to practical conditions.

All that can be said is that experiment appears to prove, what might have been predicted theoretically, that the stress due to bending in direction-fixed columns is small compared with the direct stress, provided that $\frac{L}{\kappa}$ be less than about 100. Hence the variation in the stress due to bending consequent on different end conditions does not much affect the total stress. This is the real justification for the use of experimental results as a criterion for the strength of the dissimilar practical cases.

It is, of course, needless to add that as the value of $\frac{L}{\kappa}$ increases the exact state of the end conditions becomes more and more important. Fortunately, large values of $\frac{L}{\kappa}$ are not, and should not be adopted in practice, otherwise the designer would find little real data to guide him.

THE "FREE LENGTH."—One of the most common methods of making allowance for the end conditions is to determine what is called the "free length" of the column. The deflection curve of any column is a part of a curve of sines, and by choosing a suitable part of the sine curve, the deflection curve for any end conditions can be represented.

Having thus determined a portion of the sine curve corresponding to the deflection curve, the column in question is looked upon as part of a column of which the semi-wave length is $\lambda$, and if L be the actual length of the column,

$$qL = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (424)^*$$

where $q$L is what is called the free length of the column. The coefficient $q$ depends directly on the end conditions. Thus, for an eccentrically loaded position-fixed column $q > 1$, and for an originally straight position- and direction-fixed column, if the direction-fixing be perfect, $q = \frac{1}{2}$. Every type of end condition lies between these limits.

To the extent to which the end conditions are unknown, $q$ is, of course, unknown. Supposing, however, $q$ to be determined by some means, then the

* As will be seen shortly, this equation will only hold for originally straight columns.

value of $\lambda$ can be obtained, and the column designed as a simple position-fixed column of length $\lambda$. Thus the determination of $q$ eliminates all questions of end conditions and eccentricity of loading due to whatever cause. Tetmajer, in fact, claims this as a justification for his pointed ends. But, as will be seen, the effect of initial curvature has still to be taken into account.

Many attempts have been made to determine the value of $q$. Tetmajer comes to the conclusion that for his flat-ended specimens * of wrought iron and mild steel $q = 0{\cdot}5$, for those of timber $q = 0{\cdot}5$, and for those of cast iron $q = 0{\cdot}53$. As a result of his application of the method of least squares to experiments on flat-ended specimens by various experimenters, Ostenfeld (1898) finds that the value of $q$ is only as small as 0·55 to 0·6 in *spread-out* cross sections, such as Phœnix or Z-bar columns. Under ordinary circumstances a value of $q$ varying from 0·7 to 0·8 should be taken. The fact that with flat ends the value of $q$ depends on the shape of the cross section should be borne in mind when comparing the results of experiments (see Ostenfeld's article for figures). It is, however, to be remarked that in practice no such dependence can exist. Emperger (1898 and 1908) uses the value $q = 0{\cdot}7$ for both cast-iron and mild-steel specimens with flat ends.

Several writers have proposed to express the fact that in practical end conditions the direction-fixing is imperfect by assuming a value for $q$ somewhat greater than the theoretical value for direction-fixed columns of 0·5. Thus Fidler (1887) proposes a value 0·6, and Pullen (1896) gives 0·5 to 0·63.

The values determined from experimental results are, of course, the values of $q$ at the point of failure. They by no means represent the value of $q$ in earlier stages of the experiment. From the analysis of the behaviour of a flat-ended column already given, it is evident that during the first stage of the experiment $q$ has a value much greater than 1, which is reduced during the second stage to a figure somewhat greater than one-half, and rises again after swinging round to something less than 1.

Even in direction-fixed columns the value of $q$ is not constant. It was shown in Part II, p. 71, that for working loads $\left(\frac{W}{P_2} < \frac{1}{4}\right)$ the value of $q$ in the case of perfect direction-fixing drops slowly from 0·58 to 0·56, and, except under ideal conditions, only reaches the value 0·5 when $W = P_2$ (see Fig. 15). When the direction-fixing is imperfect, if it be assumed that the free length is a mean between that for perfect direction-fixing and that for freedom in direction, the value of $q$ drops from 0·79 to 0·78 as $\frac{W}{P_2}$ varies from 0 to $\frac{1}{4}$, with a corresponding variation in the coefficient $k$ of from 1 to 1·88.

Hence, while it is incorrect to assume that $q$ is constant for any value of $\frac{W}{P_2}$ [see equation (202)], or to use a value obtained at the point of failure to calculate stresses under working conditions, it is evident that the variation in $q$ under working conditions is not large, whether the ends be perfectly fixed or not. It should be observed here that $q$ is *greater* under working conditions than at the point of failure.

It further appears that, since the value of $q$ in the ordinary column with perfectly direction-fixed ends varies from 0·58 to 0·56, the value $q = 0{\cdot}6$ is hardly sufficient to allow for the imperfections inevitable in practice, particularly in view of the distortion of the framework to which the column is connected.

* Laboratory specimens prepared with care.

For these reasons, in the absence of any more definite data, the value of $q$ in columns such as the compression members of bridges should not be taken as less than from 0·7 to 0·8. These figures, though based on quite different reasoning, agree roughly with those obtained by Ostenfeld from experiments on flat-ended specimens.

In any case a small error in the value assumed will not make much difference in relatively short members. For relatively long members it is well to err on the safe side.

Having, however, fixed the value of $q$, the procedure is not altogether so straightforward as might at first sight appear. It is evident that the value of $q$ is the same for both the perfectly straight ideal column (Case I, Variation 1) and the column with initial curvature (Case I, Variation 3). But the final shape of the column depends on its initial shape.

It is usual, having found the free length, to treat the column as an independent position-fixed column of length $q$L, and not infrequently formulæ based on eccentricity of loading have been applied to the case. Such methods are at best very rough approximations. Nevertheless, an exact method of treatment is not easy; for, in fact, the deflection of the column of length $q$L depends on the shape and conditions of the original column, and cannot be determined independently of them. This is the weakness of the method. If the original shape of the column be taken into account, the method loses its simplicity, and the equations for the direction-fixed column might be applied directly.

To illustrate the matter Fig. 51 has been drawn. This shows to scale the original and final deflections of an originally curved position- and direction-fixed column in which the direction-fixing is perfect, calculated from equation (169) Case II, Variation 3:

$$y = \frac{4\epsilon_1}{\alpha L \sin \frac{\alpha L}{2}} \left\{ \cos \alpha x - \cos \frac{\alpha L}{2} \right\}.$$

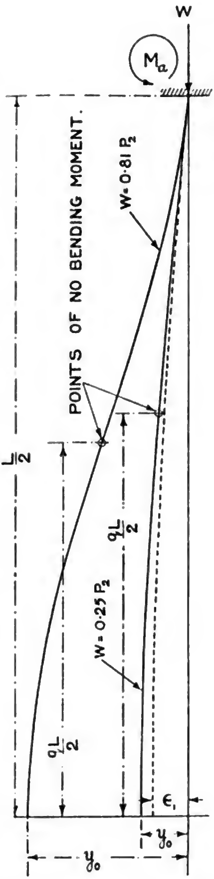



FIG. 51.

The dotted line represents the original shape of the column, the full lines its shape when W = 0·25 $P_2$ and W = 0·81 $P_2$ respectively; that is to say, its shape at the limit of working conditions, and when the load is getting somewhere near Euler's limit. The positions of the points of no bending moment are also shown. It is, of course, needless to remark that in members with an initial curvature the points of no bending moment do not coincide with the points of inflexion. From the figure it is evident

that when $W = 0{\cdot}25\ P_2$, although $q$ is greater than $\frac{1}{2}$, the bending moment at the ends is greater than that at the centre—a conclusion previously reached.

It follows from the above equation that when $\frac{dy}{dx} = 0$, $\alpha x = 0$ or $\pi$. But

$$\alpha = \frac{2\pi}{L}\sqrt{\frac{W}{P_2}}, \text{ hence } x = \frac{L}{2}\sqrt{\frac{P_2}{W}} = \lambda, \text{ if } \alpha x = \pi.$$

and the semi-wave length of the column

$$\lambda = \frac{L}{2}\sqrt{\frac{P_2}{W}},$$

which varies with W. It is also evident that the equation $qL = \lambda$ does not hold for columns having an initial curvature.

When $W = 0{\cdot}25\ P_2$, $\lambda = L$, and the point of inflexion $x = \frac{\lambda}{2}$ coincides with the point of application of the load, as is evident from the figure. It follows, therefore, that until W exceeds 0·25 P, that is up to the very limit of working conditions, the shape of the column, though fixed in direction, will be a single loop and will exhibit no points of inflexion. Not until the load passes this limit will the deflection curve assume the shape commonly associated with direction-fixed columns. In view of the fact that all practical columns have an initial curvature, these matters deserve more attention than has been paid to them.

The application of formulæ for position-fixed columns to direction-fixed columns by the process of determining a value for $q$, and hence the "free length" of the column, must, as already stated, be regarded as a rough approximation only, which leaves out of account some important factors.

It is possible, nevertheless, to determine, from the equations for position- and direction-fixed columns, formulæ based on the value of $q$ which give the stress at the centre of the column [see equations (204) and (205)].

**Effect of Form.**—From the earliest times the question of the proper shape for a column has been the subject of much debate. The strange notions held by architects in his time led Lagrange to consider the subject, and the outcome was his classic memoir (1770–3), the first scientific attempt to discover the proper shape for a column. Lagrange, adopting the Eulerian theory, came to the conclusion that the right circular cylinder was the most suitable form for a position-fixed column, and gives the *maximum maximorum* of force. This conclusion was at once challenged by others who suggested various other shapes, most of them more or less impossible.

Clausen (1851) pushed the theoretical side of the question one stage further. He assumed all the cross sections to be similar in shape, solved the general differential equation, and showed that the most suitable shape for a column is not a cylinder, for the volume of the most economical column is $\sqrt{\frac{3}{4}}$ that of the corresponding cylindrical column. Clausen does not determine the most economical shape of cross section, but he remarks that the circle is not the best shape, and Pearson (1886) has pointed out that a rectangle in which D lies between $\frac{3B}{\pi}$ and B is theoretically a better shape.

Gérard (1903) remarks that all Lagrange proved was that the right circular cylinder is the best form of truncated cone. He himself assumes the lower end of his column to be fixed in position and direction and the upper end free, and finds by an extension of Euler's method (1759) that a double truncated cone, in which the central diameter is twice that of the ends, is stronger than the right circular cylinder in the ratio of 60 to 49.

On this point the work of Kayser (1910) and Wallace (1912) may also be consulted.

Hodgkinson (1840) appears to have been the first to make experiments. He tested some cast-iron specimens in which the diameter at the centre was larger than that at the ends. Those with round ends were found to be about one-seventh stronger than uniform columns of the same weight. In the case of columns with flanged ends, no advantage was gained unless the increase in diameter at the centre was considerable. When the central diameter was half as large again as the end diameter, the column carried from one-eighth to one-ninth more than a uniform column of the same weight. Local reductions were made in the thickness of some hollow pillars by turning down the external diameter in bands. The strength of the round-ended specimens was unaffected by this, but the strength of the flat-ended specimens was reduced.

Lagrange's conclusion has, therefore, been disproved both theoretically and experimentally; nevertheless, Smith (1887) concludes that the difference between the middle and end cross sections of a column of uniform strength is in no case large, and suggests, therefore, that a uniform cross section should be used in practice.

In addition to the authors quoted above, many of the earlier writers attempted to discover the proper shape for a column of uniform strength, but their work does not appear to be of much value.

More recently, however, in connexion with the design of aeroplane struts, the question of the most economical shape for a position-fixed column of which all the cross sections are similar has again been attacked. Barling and Webb (1918), like Clausen (1851), have given a general solution to the problem, including also in their case the effect of eccentric and lateral loading. They find that the most economical column is 13 per cent. lighter than a uniform cylindrical column of the same length, which would have the same crippling load. This agrees with Clausen's result that the ratio of the volumes should be equal to $\sqrt{\frac{3}{4}}$. Webb and Lang (1919) find, however, that a column, uniform over the middle half of its length and tapering uniformly to one-half its central diameter at its ends, is 12 per cent. lighter than a uniform column of the same length and strength, and therefore only 1 per cent. heavier than the ideal strut of the best possible gradual taper. The practical advantages of such a shape are evident.

Solutions for special cases of aeroplane struts with varying cross section have been given by Berry, Case, and others.

COLUMNS WITH VARYING CROSS SECTION.—In view of the fact that the column with a varying cross section has become of practical importance in the design of aeroplanes, it may be well to review briefly the methods which have been suggested for dealing with such members.

Euler (1759) showed that in certain cases integration is possible. Lagrange (1770–3) gave a general theory applicable to such columns, but Gérard's remark regarding his final conclusion must be borne in mind. Clausen (1851)

solved the general differential equation and determined the shape of the most economical column. Winkler (1881) gave an approximate theory for columns of non-uniform cross section. Dupuy (1896–7), Francke (1901), and Wittenbauer (1902) have given analyses for columns in which sudden variations in the area of cross section occur. Dupuy considers both concentric and eccentric loading.

Duclout (1896) has suggested a simple application of the funicular polygon which might be further extended, and the graphic methods of Vianello (1898) and others might be usefully employed.

Chaudy's work (1890) is probably the most complete exposition of a method adopted by a number of writers, who replace the longitudinal load by a transverse load producing like effects. Then by an application of the principle of work an approximate expression may be obtained for the longitudinal load. Chaudy's method in brief is: replace the longitudinal load by a transverse force F acting at a distance $x$ from the end of the column. The work done by this force

$$U = c_1 F^2$$

Find the longitudinal displacement $\delta L$ of the end of the column due to the force F

$$\delta L = c_2 F^2$$

Then the minimum value of the ratio $\frac{2c_1}{c_2}$, which is a function of $x$, is the limiting value of the crippling load of the column.

The work of Engesser (1893), Gérard (1903), Kayser (1910), and Wallace (1912) may also be consulted.

More recently Bairstow and Stedman (1914) have suggested a solution of the problem by a method of building up $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, and $y$ curves, and Morley (1914) (also 1917) has proposed a method of successive approximations.

Inokuty (1907) has modified the Rankine-Gordon formula to include the case of columns of uniform strength, and Körte (1886) and Bredt (1886, 1894) have given analyses for columns of uniform curvature.

In addition to the recent work on this section of the subject quoted on p. 175, mention should be made of the approximate method of solving the differential equation given by Griffith (1919).

"Form" in the Ordinary Practical Column.—The analyses and methods reviewed above are, however, chiefly applicable to long columns to which a modified Eulerian theory will apply. Now the conditions under which the ordinary practical column (other than an aeroplane strut) exists are such that the Eulerian theory has no validity; and any theory for, or experiments on, columns with position-fixed ends have little bearing on such practical columns. Even the assumption of points of no bending moment (i.e. a value for $q$) will not help in this connexion.

The question of form may be divided into two parts, ($a$) variation in area in a longitudinal sense, ($b$) shape of cross section.

($a$) It has been pointed out that almost every practical column is imperfectly direction-fixed at its ends. Hence it follows that the points of no bending moment lie somewhere between the middle and the ends, their exact position being indeterminate. Not only so, but their position has been shown to vary

with the magnitude of the load. No exact disposition of material is possible, therefore. Further, variation in the area of cross section is only important in long columns where the bending moment is the important factor. Now, in practical columns the area is chiefly required to transmit the direct stress, and as Smith (1887) has pointed out in the remark quoted above, the variation in area is too small to be of consequence even in position-fixed columns.

Real economy is a question of cost in manufacture rather than of weight, and the saving of a few pounds of material may entail a more costly column, even though it be lighter.

Except, perhaps, in cast-iron columns and in a few special cases of timber struts, it may be laid down as a fixed rule that in practice the ordinary column position- and direction-fixed at each end should be uniform from end to end.

(*b*) The proper shape for the cross section is, nevertheless, a matter of very great practical importance. With it is bound up not only theoretical considerations as to the best disposition of the material, but questions of secondary flexure, ease in manufacture, and general convenience.

As an example of the bearing of the matter on the strength of the member, Hodgkinson's experiments (Clark, 1850), although, perhaps, extreme cases, are worth recalling. For instance, comparing experiment No. 8 of the rectangular tubes with No. 7 of the cylindrical tubes:

| Exp. No. | Weight of Tube | | Breaking Weight |
|---|---|---|---|
| 8 | Rectangular . | 82 lb. | 43,673 lb. |
| 7 | Cylindrical . . | 59 lb. | 47,212 lb. |

Examining first the theoretical expressions for the compressive stress in the ordinary position- and direction-fixed column, it has been shown that at the centre of the column, from equation (181),

$$f_c = \frac{W}{a}\left[1 + \frac{2k\epsilon_1 v_2}{\kappa^2}\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{1}{\pi^2}\cdot\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

and, at the ends of the column, from equation (184),

$$f_c = \frac{W}{a}\left[1 + \frac{2k\epsilon_1 v_1}{\kappa^2}\left\{0{\cdot}33 + 0{\cdot}29\frac{W}{P_2} - \frac{1}{\pi^2}\cdot\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

approximately. Hence, other things being equal, it follows from the above expressions that the terms $\frac{v_2}{\kappa^2}$ should be a minimum. If, further, it be assumed that $\epsilon_1$ is a function of $\kappa$, it follows that $\frac{v_2}{\kappa}$ should be a minimum. De Préaudeau (1894) and Jasinski (1894) both come to the conclusion that for eccentrically loaded columns $\frac{v_2}{\kappa}$ should be made as small as possible, and the former found that, for specimens attached by their backs, a tee section in which the flange is twice the width of the web is the most favourable cross section. Dupuy's experiments (1896) may be consulted regarding the stresses in such members. Alexander (1912) concludes that the ultimate strength of a column depends on the ratio $\frac{v_2}{\kappa}$, and that the nearer $v_2 = \kappa$ the greater the strength of the column.

The question hardly seems so simple as this, however. It is not only the stress at the centre which needs consideration. The shape of the cross section needs to be chosen so that the maximum compressive stress at both ends and middle is a minimum. It is evident that both $v_1$ and $v_2$, $k$ and $\kappa$ enter into the problem, which is further complicated by the fact that $P_2$ is a function of $\kappa$. Thus, if the expressions for the stress at the centre and ends [equations (181) and (184)] be equated,

$$\frac{v_1}{v_2} = \frac{\left\{0{\cdot}17 + 0{\cdot}26\,\frac{W}{P_2} + \frac{1}{\pi^2}\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}}{\left\{0{\cdot}33 + 0{\cdot}29\,\frac{W}{P_2} - \frac{1}{\pi^2}\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}} \quad . \quad . \quad . \quad . \quad (425)$$

from which the ratio of $v_1$ to $v_2$ for which the maximum stresses at the centre and ends are equal might be found.

In the majority of columns, however, flexure is possible in more than one plane. Hence in solid columns *isotropic* cross sections (Alexander, 1912) have an advantage, provided that the end conditions be the same for all directions. Generally an attempt is made to make $I_y = I_z$ a relic of the Eulerian theory. Kayser (1912) and Lieb would make $Z_y = Z_z$. This is equivalent to saying that $\frac{v_2}{\kappa^2}$ should be the same in both directions. It implies that both the end conditions and the initial curvature are the same in both directions, for what is required is that the stress due to bending should be equal in all directions, and can only be true for solid columns. In built-up columns, as Krohn (1908) has pointed out, quite a different set of conditions obtain, and I about the *zz* axis should be greater than I about the *yy* axis (Fig. 58).

Experiment has demonstrated that the shape of the cross section has an effect on the ultimate strength (see, for example, Fig. 70). Hodgkinson's results have already been quoted. Tetmajer (1896) remarks that in his tests on wrought-iron columns the shape of the specimen probably influenced the results. He found that riveted specimens behaved as simple rolled bars, provided that the rivet pitch did not exceed seventy times the thickness of the material, and that the rivet holes did not weaken the section by more than about 12 per cent. Föppl (1897) tested both bars weakened by rivet holes and notched bars. He found that the ultimate strength was not much affected by the rivet holes, but much reduced by the notches.

Lilly (1908) remarks that for values of $\frac{L}{\kappa}$ greater than 120 there is very little difference between the ultimate strengths in solid columns of square, circular, or other figure of cross section. For values of $\frac{L}{\kappa}$ less than 120 and greater than 30 the shape of the figure of the cross section influences the strength, and the values obtained are somewhat less than those for a circular cross section. This conclusion, as Engesser and others have pointed out, follows from the Considère-Engesser theory, though Engesser (1895), Kármán (1910), and Southwell (1912) agree that the influence is not great.

Christie (1884) remarks that short lengths of channel offered less resistance than corresponding lengths of angle or tee sections with equal radii of gyration, due, he considers, to the greater extent of unbraced web in the channel. Channels showed local failure or crippling rather than bending when $\frac{L}{\kappa}$ was as

high as 37, whereas with angles and tees no such failure occurs when $\frac{L}{\kappa}$ is higher than 30. Similar phenomena were observed in the channels tested at Watertown Arsenal (1882–5).

SECONDARY FLEXURE.—The effect of secondary flexure or wrinkling is evident in the experiments referred to above. It is also plainly visible in the shorter tubes tested by Hodgkinson (Clark, 1850), in the 1908 Watertown Arsenal experiments, and wherever, in fact, thin tubes have been tested as columns.

Box (1883) was the first to propose a theory for wrinkling. He suggests the formula

$$f_w = 80\sqrt{\frac{t}{\Omega}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (426)$$

where $f_w$ = the load in tons per square inch producing wrinkling.
$t$ = the thickness of the plate in inches.
$\Omega$ = the unsupported length of the plate in inches (see Fig. 52).

More recently Lilly (1905–7) has devoted considerable attention to this aspect of the subject. He tested a large number of thin tubes of mild steel and found that when $\frac{L}{\kappa}$ was less than 80 for the thinnest tubes, wave formation or wrinkling was set up, so that the strength of the wave to resist compression is the true compressive strength of the column. He finds that the wave length $2\lambda$ varies as the square root of the area of the cross section. Actually

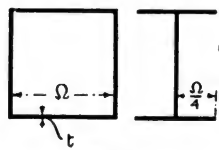


FIG. 52.

$$\lambda = \frac{\pi}{\sqrt[4]{12}}\sqrt{rt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (427)$$

and

$$f_w = \frac{f_o}{1 + c \cdot \frac{\kappa}{t}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (428)$$

where $r$ is the radius of the tube and $c$ is a constant $= \frac{1}{8}$ for hollow circular columns of mild steel. To combine the effects of both primary and secondary flexure he uses a formula which may be written

$$f_r = \frac{f_c}{1 + \frac{Kf_o}{E} \cdot \frac{\kappa}{t} + \frac{f_o}{4\pi^2 E} \cdot \frac{L^2}{\kappa^2}} \quad . \quad . \quad . \quad . \quad . \quad (429)$$

The formula as it stands applies to position- and direction-fixed ends; if the ends be merely position-fixed, the 4 in the denominator disappears. The values given for K are

| | |
|---|---|
| Circular cross section . . . | K = 50 |
| Square ,, ,, . . . . | K = 60 |
| Cruciform ,, ,, . . . . | K = 120 |
| Triangular ,, ,, . . . . | K = 80 |
| Beam ,, ,, . . . . | K = 70 |

R. Earle Anderson (1911) determines a limiting thickness for short tubular columns from Lilly's formula (428) thus: The formula may be written

$$\frac{\kappa}{t} = \frac{1}{c}\left(\frac{f_c}{f_w} - 1\right)$$

where $c = \frac{1}{8}$. If now it be assumed that $f_w = f_e = f_c \div 2$, then $\frac{\kappa}{t} = 8$. Alternatively, if $f_w = f_e = 30{,}000$ lb. sq. in., and $f_c = 80{,}000$ lb. sq. in. (Lilly's figure for mild-steel columns), then $\frac{\kappa}{t} = 13$. Taking the average of these two values and calling $\frac{\kappa}{t} = 10$, it follows that the limiting value of $\frac{r}{t} = 14{\cdot}1$, or the thickness of tubular columns should not be less than one-thirtieth of the diameter. This value applies, of course, to columns one wave length long.

It will be seen that according to the theories of Box and Lilly the resistance to wrinkling depends on two different conditions.

According to Box it is the unsupported width of plate which is important. Bouscaren (1880), from the results of his experiments, concludes that the thickness of metal should not be less than one-thirtieth the distance between the supports transversely. That is to say, $\frac{t}{\Omega} = \frac{1}{30}$. Substituting this in Box's formula (426),

$$f_w = 80\sqrt{\frac{1}{30}} = 14{\cdot}6 \text{ tons sq. in.,}$$

a stress not very different from the yield point of the material (wrought iron).

According to Lilly it is the wave length longitudinally on which the resistance to wrinkling depends. This may or may not be influenced by the value of $\frac{t}{\Omega}$, but no direct allowance for such influence is made in Lilly's formula.

Roark (1913, 1916), as the result of his analysis and experiments on the wrinkling of an outstanding flange, gives the formula

$$f_w = 0{\cdot}6\ \frac{Et^2}{B}$$

where B is the breadth of the outstanding flange.

The subject appears to need further study. What is the exact influence of transverse support, to what extent does this support affect the longitudinal wave length? Is the wave length longitudinally what might be called the natural wave length of the flange plates, or a forced wave length equal to the distance between the centres of the rivets, or some combination of the two? What is the exact influence of variations in the distance apart of the points of support longitudinally and transversely on the strength of the column? These appear to the author to be questions which cannot be answered in the present state of knowledge.

Since the above was written, the question of wrinkling in short hollow tubular struts has received considerable attention, particularly in view of the employment of thin hollow tubes for aeroplane struts. By his general analysis Southwell (1914) has confirmed (for practical purposes) Lilly's (1907) and Lorenz's (1908) formulæ for ring-wave deformation, and extended his analysis to cover lobe-form deformation, though his results appear to differ from those

of Lorenz (1911) even if the effect of end conditions and length which the latter has included in his analysis be eliminated. Southwell's equation for the load per unit area producing wrinkling reduces to

$$f_w = \frac{Et}{r}\sqrt{\frac{1}{3} \cdot \frac{m^2}{m^2+1}} \cdot \frac{k^2-1}{k^2+1}$$

where $r$ = the mean radius of the tube,
$t$ = its thickness,
$k$ = the number of lobes in the distorted form of the cross section,
$m$ = Poisson's ratio $= \frac{10}{3}$.

For ring-wave deformation $k = 0$. When the deformation is lobe-formed, $k = 2$ or $3$. When $k = 1$ the formula becomes

$$f_w = \frac{\pi^2 E r^2}{2\lambda^2}$$

where $\lambda$ is the semi-wave length, and corresponds to Euler's formula for a tube of length $\lambda$.

Experimental work has been done by W. H. Barling on mild steel tubes, and by Popplewell and Carrington (1917) on high-tensile steel tubes, both hard and annealed. It would appear from this that in the case of the mild-steel and the annealed tubes, there is a critical ratio of thickness to radius, approximately $\frac{t}{r} = 0{\cdot}1$, below which there is a definite wrinkling stress, which varies approximately with $\frac{t}{r}$, and above which the elastic breakdown coincides with the elastic limit. Popplewell and Carrington report that when $\frac{t}{r} > 0{\cdot}1$ the wrinkles were circular; for values of $\frac{t}{r} = 0{\cdot}1$ and slightly less, the distorted shape was oval or two-lobed, corresponding to Southwell's $k = 2$. When $\frac{t}{r}$ = about 0·05 three lobes ($k = 3$) appeared. They remark, however, that the measured wave lengths were from 30 to 100 per cent. greater than the critical values given by Lilly's and Southwell's formulæ.

According to Robertson (1920), however, for tubes of ductile material in which the elastic limit and yield point are nearly identical, when $\frac{t}{r}$ is greater than 0·006 yield precedes collapse by wrinkling. When $\frac{t}{r}$ is greater than approximately 0·044 complete collapse occurs at higher stresses than the yield; whilst thinner tubes sustain the yield stress, and collapse immediately by the walls "caving in." Robertson points out that Southwell's formula (ring-wave deformation) would not apply to mild-steel tubes in which $\frac{t}{r} > \frac{1}{400}$ if the elastic limit of the material be 20 tons sq. in. Such thin tubes would not be found in practice.

The question of wrinkling in short tubular struts concentrically loaded is, however, only a part of the question at issue. In long practical struts a combination of primary and secondary flexure (wrinkling) occurs. This is plainly visible in Hodgkinson's experiments on long thin tubes (Clark, 1850) where the wrinkling appears only on the most compressed half of the circumference; for example, in flat-ended specimens, on opposite sides of the tube at the ends and middle. A complete theory should take both primary and secondary flexure into account.

In connexion with the subject of secondary flexure the question of *shape deformation* has not been given the consideration it deserves. Many experimenters have recorded the fact of the distortion of the shape of the cross section, with consequent reduction in the strength of the specimen. Recently the experiments of the Column Committee of the American Society of Civil

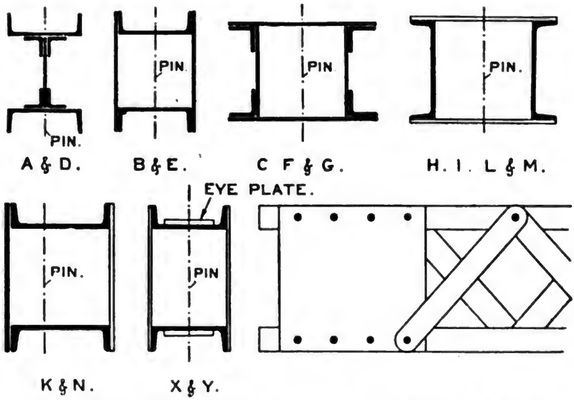



Fig. 53.—**Experiments on Built-up Wrought-iron Columns (Watertown Arsenal, 1882–5).**

Engineers have again called attention to this phenomenon (*Engineering News-Record*, New York, June 28, 1917, p. 640; Feb. 7, 1918, p. 250).

Some information regarding secondary flexure in compression members will be found in A. P. Thurston's paper (1919). For the very thin material used in the metal spars of aeroplanes he finds that the ultimate strength of a plain angle section of breadth B and thickness $t$, when $\frac{B}{t} = 30$, is 7·8 tons sq. in. If $\frac{B}{t} = 7$, the limiting stress is 17·5 tons sq. in. The most economical value for $\frac{B}{t}$ appears to be from 6 to 8, and should not exceed 10. To prevent local buckling the pitch of the rivets should not exceed 15 $t$.

Practical Cross Sections.—There is probably a theoretically most economical thickness for any shape of cross section, and Lilly, in fact, determines a value for this by equating the resistance of primary to secondary flexure. The

engineer can in ordinary practical columns seldom adopt such an economic thickness—questions of rusting and practical considerations determine a

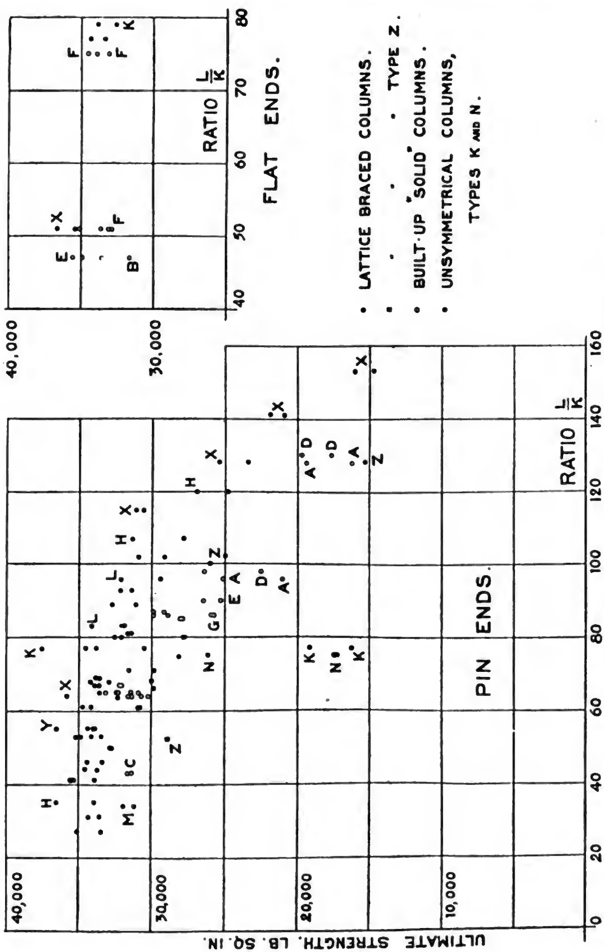



FIG. 54.—Built-up Wrought-iron Columns (Watertown Arsenal, 1882–5).

minimum thickness, and the problem resolves itself rather into that of flexure between the rivets in the flanges, already considered in the section on built-up

columns (p. 107). To him the question is how best to arrange plates and rolled sections in order to produce a convenient and economical column. Certain types suitable for given conditions have become standardized by experience, and are too well known to require description. For small bridge members the double channels connected by lattice bars (Fig. 31) is the most common. In this connexion the long series of experiments made at Watertown Arsenal (1882–5), the results of which are plotted on Fig. 54, are worth consideration. From an examination of this figure the somewhat surprising fact appears that the built-up "solid" columns are weaker than the lattice-braced members. This is due, no doubt, particularly in the case of the smaller values of $\frac{L}{\kappa}$, to the flange plates in the built-up "solid" columns buckling between the rivets. This suggestion will not explain the low positions of types A and D (see Fig. 53), where the metal is evidently not disposed to the greatest advantage. The best results were obtained from the common type H, L, X, and Y (Fig. 53). In drawing conclusions from these experiments the eccentrically loaded unsymmetrical sections K and N should be ruled out, together with type Z, in which the imperfections were abnormally large.

On the other hand, as far as can be judged owing to the different end conditions, Strobel's Zed-iron columns (1888) appear to be quite as efficient as the common double channel type, although they approximate in shape rather to types A and D. On the whole it appears probable that the common form is as advantageous as any for ordinary bridge members.*

When the size of such members is much increased, however, the question of form becomes more important. Attention has been directed to this question by the failure of the Quebec Bridge.

In the Forth Bridge the large compression members were of the form of large tubes stiffened by internal radial ribs. This type is somewhat costly, and does not lend itself to simple attachments. The more modern practice is to use a number of parallel flanges connected by web bracing and diaphragm plates, the whole forming a rectangular section (Figs. 55 and 56). A collection of this type of section as used in large American bridges will be found in Appendix 17, *Report of the Quebec Bridge Commission*, *Engineering News*, New York, April 30, 1908, and as used in German bridges in *Der Eisenbau*, Leipzig, March 1914, No. 3, pp. 109 and 110.

The failure of the chord in the Quebec Bridge was due to inadequate web bracing, but even if the bracing be made sufficiently strong such sections are not ideal. A box section would in many respects be preferable, but for the difficulty of painting. On the whole, a compromise of the type suggested by Hodge (1913) and shown in Fig 55 seems to be the most suitable form. The plate web ensures rigidity and resistance to shear, whilst the flange material can be arranged to resist the bending moment in both directions. There are no practical difficulties in the construction, the member can be properly protected against rusting, and attachments are not difficult.

**The Built-up Column.**—A very large proportion of the columns used in practice are built up of separate sections riveted together. The sections may be joined together directly, as, for example, two angles riveted back to back to

* See the recent experiments of the Column Committee of the American Society of Civil Engineers (*Engineering News-Record*, N.Y., June 28, 1917, p. 639), which appear to have shown that the effect of form in well-designed columns is not considerable.

form a tee, or the colûmn may be constructed girder fashion with two or more flanges united by a web or webs. It is usually taken for granted that such columns behave as solid specimens provided the web and the riveting be of sufficient strength, and formulæ for solid columns are commonly applied to them. It is further quite common to find in text- and pocket-books rules and tables designed with the object of making the moment of inertia of the built-up column equal in the direction of both principal axes, although it is recognized that in some columns of this type the weakest part may be the flange acting as a column between the panel points of the web bracing.

Now if one point be clearer than any other with regard to built-up columns, it is that they do not act as solid or homogeneous columns. Unless improperly designed they always fail locally by the flange buckling between the panel points, and it is their strength in this connexion which determines their strength as a whole.* Hence it follows that all rules and deductions based on assumptions of solidity or homogeneity are beside the question.

Two noteworthy experimental analyses of the strains in built-up columns have been published in America, one by Talbot and Moore (1909), the second by Howard (1911). In drawing deductions from the first it should be borne in mind that the specimens were abnormal. The majority were old compression members cut out of a bridge and subjected without doubt to severe treatment during the process. The new specimen was intentionally built of thin material. Nevertheless, the results, though for this reason probably exaggerated, are sufficiently remarkable.

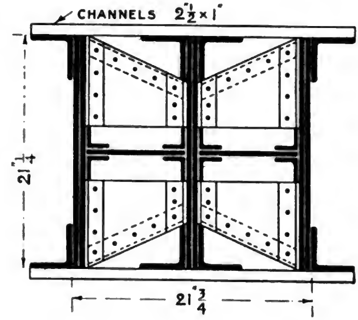



FIG. 55.

It was found that the stress varied considerably over the length of the specimens, making it doubtful whether the component parts of a built-up column act together to form an integral compression member. There were indications of stresses in the extreme fibres from 40 to 50 per cent. in excess of the average stress, and in some cases even higher. In a compression member of an actual bridge the maximum stress was 73 per cent. in excess of the mean. The maximum stress would occur at one cross section in the extreme fibres on one side of the channel, at a near-by section on the other side. These irregularities appear to have been due to local flexure, for the authors record that the channels forming the flanges showed evidence of considerable local flexure, due apparently to initial want of straightness. They conclude, in fact, that want of straightness of the centre line of the column and eccentricity of loading may have much less effect than local want of straightness. In an actual bridge secondary bending and twisting due to a cross girder may even have more effect than any of the above imperfections. In a U-shaped girder flange the addition of the top or bottom plate tends to reduce the local irregularities, and the stresses were

* If, of course, the column deflect in a direction perpendicular to the plane of the web bracing only, it will act as a solid column, and no question of built-up columns arises.

found to be more uniform. No relation was found between the actual stresses and the stresses computed by column formulæ. The distribution of stress under working loads may be quite different from that when the column becomes crippled, consequent on the redistribution of stress after the yield point has been reached in some of the fibres.

In the second experimental study of the stresses in a built-up column by Howard and Buchanan (1911), the specimens were more normal (Fig. 56). Careful measurements of the strain in various parts of the column were made. The ultimate strength of the specimens was 30,490 lb. sq. in. For a range of load from 220 to 8,817 lb. sq. in. the longitudinal contraction, measured in a gauged length of 150 in. along the middle of the flange plate, was 0·0408 in. The corresponding readings at the edges were 0·0394 and 0·0410 in., the mean of which is 0·0006 less than the observed contraction of the centre gauged length. The readings at the opposite corners were 0·0467 and 0·0485 in. respectively, the mean for all four corners being 0·0439 in. The contractions in local gauged lengths of 10 in. were measured at different parts of the column. On the flanges these contractions varied from 0·0026 to 0·0033 in.

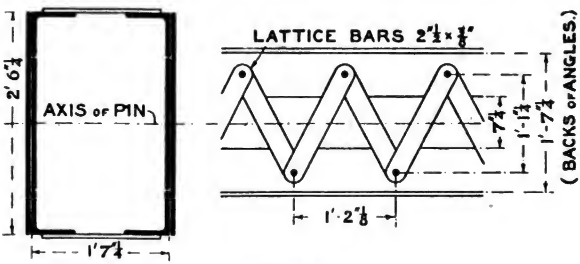



Fig. 56.

In stepping from the pin plates to the flange plates, the contractions varied from 0·0033 to 0·0043 in., indicating a movement of the pin plates along the column. On the pin plates themselves the maximum contraction was observed directly in front of the pin. Abreast of this place, toward the edges of the plate, the stress became tensile (Fig. 57).

Assuming the ratio of lateral extension to longitudinal contraction to be 1 : 3·55, the lateral extension corresponding to the actual longitudinal compression should have been 0·0023. On the flange plates the observed lateral extension varied from 0·0019 to 0·0026 in. On the lattice side the extension in the overall width was 0·0015 in., between the rivet centres 0·0012 in. (Fig. 56). These measurements were made midway between the diaphragm plates. At the diaphragm plates the lateral extension was only 0·0001 in.

Up to a certain load (12,000 to 20,000 lb. sq. in. in different columns) the stress-strain diagram for the column followed the line $E = 29{,}500{,}000$, and then began to fall away. This was owing to the gradual development of permanent sets due to local imperfections, which imperfections appear, however, to have relatively less effect in large than in small columns. Permanent yielding takes place some time prior to reaching the maximum load. Time is a factor in determining the latter. The full contraction is not reached immediately, and

the apparent ultimate strength would doubtless be lowered by prolonged loading. An ultimate strength coincident with the elastic limit in individual members should be attained, whilst from the ease with which local buckling then takes place no higher resistance will be realized or should be expected.

Similar measurements of the contractions in gauged lengths at various points in the length of columns will be found in the Watertown Arsenal Reports from 1910 onward. Howard (1909) remarks regarding some built-up beam sections tested at Watertown Arsenal: "The short built-up columns did not display the increase in strength observed in tubes and rolled sections. The plates and angles of these short built-up members appear to act independently, rather than as parts of the whole."

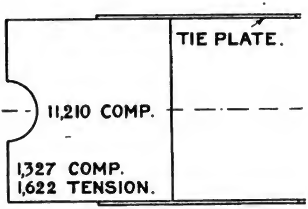



FIG. 57.

It is, in fact, as all experiments show, the local strength to resist buckling of the individual parts which determines the strength of built-up columns, and if the web be sufficiently strong to resist the shearing force, the built-up column always fails by the flange crippling between the panel points.

Many writers have come to this conclusion, but it remained for Krohn (1908) to give the first rational theory for such columns. Krohn's analysis is open to the objection that he assumes the constants in Tetmajer's straight-line formula to represent the actual stresses in the material. Nevertheless his fundamental ideas are undoubtedly right. His reasoning is this: For a given deflection in the plane of the webs the load in the concave flange of the column will be greater than one-half the load W by an amount which can be found in terms of the deflection. Hence the load on one of the elementary columns into which the flange is divided by the bracing may be discovered. The strength of this elementary column to resist this load determines the ultimate resistance of the column as a whole. The only difficulty is to find the deflection. Krohn determines a value for it in terms of the constants in Tetmajer's formula, which leads to a simple formula for built-up specimens by which experimental results may be interpreted with success. From this formula it would appear that in very short columns $F_2$, the load in the concave flange, approaches the limit $0{\cdot}5$ W. When $\frac{L}{\kappa} = 105$, the validity limit of Tetmajer's straight line, and hence of Krohn's formula, $F_2 = 0{\cdot}81$ W. Working backward, it is evident that the formula will determine the maximum length of the elementary flange columns, that is to say, the pitch of the panel points. Krohn shows that if the elementary flange columns are to have the same factor of safety as the column as a whole deflecting about the *zz* axis (Fig. 58), the pitch of the panel points must be zero. In short, the column will be solid. Hence the I of the column about the *zz* axis should always be *greater* than that about the *yy* axis, and the unsupported length of the flanges should be so chosen that their factor of safety against crippling is equal to that of the column as a whole about the *yy* axis.

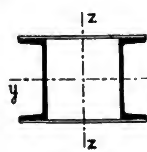



FIG. 58.

A criticism and modification of Krohn's formula by Engesser (1909), an

extension by Saliger (1912), and a generalized form of the analysis by Gérard (1913) should be considered in this connexion. Krohn's formula applies particularly to mild-steel specimens, but Schaller (1912) has modified the constants to suit nickel-steel columns. Brik (1911) raises a point which is worth noticing. Krohn in his analysis has assumed that the radius of gyration of the column as a whole is equal to $\frac{h}{2}$. This is practically true for normal cross sections; but when the distance between the flanges is small, the value of $\kappa$ may differ considerably from this value, and the formula give erroneous results in consequence (see p. 272 for Krohn's formula).

In addition to Krohn's theory and its variations, a much more elaborate theory for built-up columns has been given by Müller-Breslau (1910–11). A somewhat similar analysis for concentrically loaded columns is due to Mann (1909), and Grüning (1913) has extended the Müller-Breslau analysis and applied to it the Considère-Engesser theory.

In the Müller-Breslau analysis account is taken of the deformation of the panels due to shear, the consequent bending moments at the panel points, and also the deformation in the web bracing. The result is a complicated set of formulæ which apply only to long columns. Reduction coefficients are introduced to simplify the expressions, and for short columns the equations are modified to suit Tetmajer's straight line for solid columns. The analysis appears to be rather complicated for practical use and applicable rather to abnormal cases where the deformations considered may have an appreciable effect on the strength of the column. In brief, Müller-Breslau finds that the expression for the crippling load of a long lattice-braced column takes the form of Euler's equation, but I is replaced by a function depending on the area and moment of inertia of the component parts. He finds that eccentricity of loading has no great effect on the crippling load, but considerably affects the stresses set up. In symbols, the crippling load for a long column is

$$R = \frac{\pi^2 EI}{L^2} \xi_1 \xi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (430)$$

where $\xi_1$ and $\xi_2$ are coefficients depending on the strength of the bracing and the number and arrangement of the panels. If $\frac{L}{\kappa}$ be small, and the material has passed the elastic limit, this equation is written

$$R = \xi_1 \xi_2 \left(3{\cdot}1 - 0{\cdot}0114 \frac{L}{\kappa}\right) a \quad t/cm^2 \quad . \quad . \quad . \quad . \quad (431)$$

The analyses of Engesser (1891), Prandtl (1907), Kayser (1910), and others may be looked upon as less complete forms of the Müller-Breslau analysis. The chief result of these complicated studies, which include the effect of stresses in the diagonals, is to confirm theoretically what has also been proved experimentally, that the effect of the shearing force on the ultimate strength of well-designed columns is, practically speaking, negligible.

This being the case, the far simpler analysis for built-up columns given in Part II, p. 91, which includes the Krohn effect and endeavours to avoid the objections to his analysis, seems all-sufficient for practical work.

THE WEB SYSTEM.—Although the effect of the shearing force on the crippling load of a column is small, it is quite possible for a built-up column to fail due merely to the weakness of the web system, as, in fact, did the model of the chord which failed in the Quebec Bridge. Nevertheless, direct measurement of the stresses in the lattice bars under ordinary conditions proves them to be very small. In the Watertown Arsenal Report, 1909–10, will be found an account of the measured contractions in a length of six inches on each of 33 lattice bars of a built-up column. The specimen failed at 33,000 lb. sq. in. Up to 10,000 lb. sq. in. the contraction was zero in all bars. At 15,000 lb. sq. in. the contraction was 0·0001 in three bars, and in the rest zero. At 20,000 lb. sq. in. the contraction was 0·0001 in four bars, 0·0002 in one bar, and in the rest zero. On removal of the load these strains disappeared.

Howard and Buchanan (1911), as part of their analysis of the stresses in built-up columns, measured the strains in the lattice bars. The ultimate strength of the members was 30,490 lb. sq. in. At a load of 8,597 lb. sq. in. the strains in the lattice bars were so small that they could not be detected. By combining the longitudinal contraction with the lateral extension it was found that the length of the lattice bars remained, practically speaking, constant, thus confirming the stress observation. Even when the load was increased to 28,667 lb. sq. in., an extension of 0·00002 (?) in. only was observed in a lattice bar, corresponding to a stress of 600 lb. sq. in.

Talbot and Moore (1909) estimate the stresses in the lattice bars of their experimental columns as equivalent to a transverse shear of 1 to 3 per cent. of the longitudinal load ($Q = 1$ to 3 per cent. of W). In an actual bridge, however, the stresses were so small that they could not be measured. These authors remark that the usual form of lattice bar is a very inefficient compression member when eccentrically loaded through a riveted connexion. Under ordinary circumstances the maximum stress may be as much as three times the average stress. In tests of single bars which they made, the ultimate strength was always less than one-half the elastic limit. To represent the ultimate strength they propose the straight-line law

$$f_r = 21{,}400 - 45\frac{L}{\kappa} \text{ lb. sq. in.}$$

That the shear stress in the lattice bracing is small is the conclusion to which all who have attacked the problem have come. Fidler (1887) remarks that the greatest theoretical stress will often be so light that the theoretical section must be largely increased in practice. Engesser (1891) determined an expression for the magnitude of the shearing force thus: The maximum stress in the flanges

$$f_c = \frac{W}{2a_2} + \frac{W\Delta}{a_2 h}.$$

Hence

$$\Delta = \left(\frac{a_2 f_c}{W} - \frac{1}{2}\right) h$$

$$Q = \frac{dM}{dx} = W\frac{dy}{dx} = -\frac{\pi}{L}\sin\frac{\pi x}{L}\left(a_2 f_c - \frac{W}{2}\right) h.$$

This is a maximum when $x = \frac{L}{2}$,

$$Q_{max} = \frac{\pi h}{L}\left(a_2 f_c - \frac{W}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (433)$$

This formula gives the maximum shearing force in terms of $f_c$. Engesser proposes to make the bracing and the column as a whole reach the crippling load at the same moment, to do which $f_c$ must be given its value at the moment of failure; but this value is unknown.

Keelhof (1893) determined the value of $y_0$ from the Rankine-Gordon formula, and thus obtains a value for Q:

$$y_0 = c_2 \frac{L^2}{\kappa},$$

$$y = y_0 \sin \frac{\pi x}{L} = c_2 \frac{L^2}{\kappa} \sin \frac{\pi x}{L}.$$

Hence $$Q = \frac{dM}{dx} = c_2 \frac{L}{\kappa} W \cos \frac{\pi x}{L}$$

and its maximum value $$Q_{max} = c_2 \pi \frac{L}{\kappa} W \quad \ldots \ldots \quad (434)$$

In 1907 he modified the analysis to suit Tetmajer's straight line. The maximum value of Q may be written

$$Q_{max} = \frac{\pi \kappa^2}{v_2 L} \left( f_c a - W \right) \quad \ldots \ldots \quad (435)$$

[compare Engesser's equation (433) above]. If now the straight line

$$f_r = c_1 - c_2 \frac{L}{\kappa}$$

be identical with $$f_a = f_c - f_b,$$

then $$(f_c a - W) = c_2 a \frac{L}{\kappa}$$

and equation (435) becomes

$$Q_{max} = \pi c_2 a \frac{\kappa}{v_2} \quad \ldots \ldots \quad (436)$$

In these formulæ, if $c_2$ be taken from an ultimate strength formula, Q must be the shearing force at the point of failure. Jensen (1908) criticizes this value for Q on the ground that $c_1$ in Tetmajer's straight-line formula is not the stress in the material. His remarks are worthy of attention. He points out that the shearing force is much greater if the eccentricity at the two ends lie on opposite sides of the central axis. In this case the maximum value of the shearing force occurs at the centre, and is

$$Q_{max} = \frac{\frac{L}{2\kappa}\sqrt{\frac{f_a}{E}}}{\sin \frac{L}{2\kappa}\sqrt{\frac{f_a}{E}}} \frac{2\epsilon_2 W}{L} \quad \ldots \ldots \quad (437)$$

If the eccentricity at the two ends of the column lie on the same side of the

central axis, the maximum value of the shearing force will occur at the ends, and is

$$Q_{max} = -\frac{W\epsilon_2}{\kappa}\sqrt{\frac{W}{Ea}}\tan\frac{L}{2\kappa}\sqrt{\frac{W}{Ea}} \quad . \quad . \quad . \quad (438)$$

(*Engineering*, London, September 1907, p. 402).

These two values for Q are rational values, and apply within the elastic limit.

In 1905 L. Vianello gave another expression for the shearing force Q in mild-steel columns, determined from Tetmajer's straight line. He assumed that the term $c_2 \cdot \frac{L}{\kappa}$ in the formula was the stress due to bending, found an equivalent uniform lateral load which would produce such a stress, and hence obtained an expression for the shearing force. He found that

$$Q_{max} = \frac{a_2}{11} \text{ metric tons,}$$

which is the value of the shearing force at the point of failure. If a factor of safety of 3 be allowed, the working value for Q is

$$Q_{max} = \frac{a_2}{33} \text{ metric tons.}$$

$a_2$ is the area of one flange in square centimetres.

A somewhat more consistent analysis has been given by Krohn (1908) and others. Krohn assumed that the curvature was sinusoidal, in which case the maximum shearing force, which occurs at the ends, is

$$Q_{max} = Wy_0\frac{\pi}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (439)$$

Assuming from Tetmajer's straight-line formula for mild-steel columns that

$$f_b = \frac{Wy_0}{Z} = 0{\cdot}0114\,\frac{L}{\kappa}$$

it follows that

$$Q_{max} = \frac{Z}{28\kappa} = \frac{a}{28}\cdot\frac{\omega}{\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (440)$$

which is Keelhof's formula (436). If now the two flanges be equal in area, $2a_2 = a$, $Z = a_2h$, and $\kappa = \frac{h}{2}$.

Hence

$$Q_{max} = \frac{a_2}{14} = \frac{a}{28} \text{ metric tons} \quad . \quad . \quad . \quad . \quad (441)$$

As before, the areas $a_2$ and $a$ are in cm². The formula gives the shearing force at the point of failure.

Several writers have proposed to determine the shearing force by assuming that it is equal to the shearing force in a beam so loaded that the bending moment at its centre is equal to that at the centre of the column. It is difficult to see what is gained by making such an assumption.

It would appear from the analyses which have already been given for

determining the shearing force in a column that the effect of initial curvature in augmenting the shearing force has not been generally recognized. In an ordinary column with position-fixed ends (Part II, Case I, Variation 6) the bending moment anywhere, from equation (84), is

$$M = Wy = W\left\{\left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}\cos ax - \frac{8\epsilon_1}{a^2L^2}\right\}$$

from which

$$Q = \frac{dM}{dx} = -aW\left\{\left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)\sec\frac{aL}{2}\sin ax\right\} \quad . \quad . \quad . \quad (442)$$

The value of Q is a maximum when $x = \frac{L}{2}$. Neglecting the minus sign,

$$Q_{max} = W\left\{\left(\epsilon_2 + \frac{8\epsilon_1}{a^2L^2}\right)a\tan\frac{aL}{2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (443)$$

or, with the usual approximations,

$$Q_{max} = \frac{2W}{L}\left\{\left(\epsilon_2 + \frac{8P\epsilon_1}{\pi^2W}\right)\frac{\pi}{2}\sqrt{\frac{W}{P}}\tan\frac{\pi}{2}\sqrt{\frac{W}{P}}\right\} \quad . \quad . \quad (444)$$

At the limit of working conditions, $\frac{W}{P} = \frac{1}{4}$, and

$$Q_{max} = \frac{\pi W}{2L}\left(\epsilon_2 + \frac{32\epsilon_1}{\pi^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (445)$$

It is evident from this equation that under these conditions, if the value of the initial deflection $\epsilon_1$ be equal to the eccentricity $\epsilon_2$, the effect of $\epsilon_1$ in causing shearing forces in the column will be more than three times as great as that of $\epsilon_2$.

Giving to $\epsilon_1$ the value $\frac{L}{750}$ and to $_2$ the value $\frac{L}{1000}$, the value of $Q_{max}$ becomes

$$Q_{max} = 0{\cdot}0082\ W.$$

That is to say, under working conditions the maximum shearing force in an ordinary position-fixed column is at least 1 per cent.* of the longitudinal load W (compare Talbot and Moore, 1909).

It is of interest to compare this with Jensen's equation (437) for the maximum shearing force when the eccentricity at the two ends lies on opposite sides of the central axis. That equation may be written

$$Q_{max} = \frac{2\epsilon_2 W}{L}\frac{\pi}{2}\sqrt{\frac{W}{P}}\operatorname{cosec}\frac{\pi}{2}\sqrt{\frac{W}{P}}.$$

Giving to $\epsilon_2$ the same value as before, $Q_{max}$ at the limit of working conditions when $\frac{W}{P} = \frac{1}{4}$ is

$$Q_{max} = 0{\cdot}0022\ W,$$

* If the eccentricity due to variations in the modulus of elasticity be taken into account, this figure will of course be increased.

which is considerably less than the above, but $\sqrt{2}$ times the value of $Q_{max}$ from equation (445) were $\epsilon_1$ equal to zero.

Considering next the case of an ordinary column with position- and direction-fixed ends (Part II, Case II, Variation 7), the deflection of the column is given by equation (169)

$$y = \frac{4k\epsilon_1}{aL}\left(\frac{\cos ax - \cos\frac{aL}{2}}{\sin\frac{aL}{2}}\right) + \epsilon_2.$$

The bending moment anywhere, from equation (200), is

$$M = Wy + M_a = \frac{4Wk\epsilon_1}{aL}\left(\frac{\cos ax - \cos\frac{aL}{2}}{\sin\frac{aL}{2}}\right) - \frac{8\ W\epsilon_1}{a^2L^2}\left(1 - k\frac{aL}{2}\cot\frac{aL}{2}\right)$$

from which $$Q = \frac{dM}{dx} = -\frac{8W\epsilon_1}{a^2L^2}\left(ak\frac{aL}{2}\operatorname{cosec}\frac{aL}{2}\sin ax\right) \quad . \quad . \quad . \quad (446)$$

and $$\frac{d^2M}{dx^2} = -\frac{8W\epsilon_1}{a^2L^2}\left(a^2k\frac{aL}{2}\operatorname{cosec}\frac{aL}{2}\cos ax\right)$$

Hence it follows that Q is a maximum when $ax = \frac{\pi}{2}$ or when

$$x = \frac{L}{4}\sqrt{\frac{P_2}{W}}$$

since $a = \frac{2\pi}{L}\sqrt{\frac{W}{P_2}}$ approximately.

If $\frac{W}{P_2} = \frac{1}{4}$, $x = \frac{L}{2}$, and the maximum shearing force occurs at the ends of the column. When $\frac{W}{P_2}$ is less than $\frac{1}{4}$, that is to say, under all working conditions, $x$ is greater than $\frac{L}{2}$, and the point of inflexion, $\frac{d^2M}{dx^2} = 0$, lies outside the column (compare Fig. 51 and the remarks thereon). The shearing force will still be a maximum at the ends of the column. When $\frac{W}{P_2}$ is greater than $\frac{1}{4}$, the points of inflexion will lie inside the column and the shearing force will no longer be a maximum at the ends.

Limiting this enquiry to working conditions, $\frac{W}{P_2} < \frac{1}{4}$, $x$ may be put equal to $\frac{L}{2}$ in order to obtain the maximum shearing force, which then becomes, from equation (446),

$$Q_{max} = W\frac{4k\epsilon_1}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (447)$$

Since the column is fixed in direction at the ends, this expression might have been obtained directly from equation (163).

Giving to $\epsilon_1$ the value $\frac{L}{750}$, and to $k$ the value 1·88 from Fig. 15 corresponding to the limit $\frac{W}{P_2} = \frac{1}{4}$ under the conditions there assumed,

$$Q_{max} = 0{\cdot}010\ W$$

That is to say, under working conditions the maximum shearing force in an ordinary position- and direction-fixed column will be at least 1 per cent.* of the longitudinal load W, or about the same percentage as in the case of the position-fixed column.

It is possible, as Hutt (1912) has pointed out, that the shearing force in a direction-fixed column may be increased due to the general distortion of the structure of which it forms part. That writer estimates the value of the shearing force at the ends of a position-fixed column to be

$$Q_{max} = \frac{\pi f_b I}{L v_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (448)$$

and points out that at the centre it may possibly reach one-half this value. In the case of position- and direction-fixed columns he concludes that the maximum shearing force occurs at the points of contraflexure, and is approximately

$$Q_{max} = \frac{1{\cdot}25\pi f_b I}{L v_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (449)$$

He estimates the additional shearing force due to distortion of the panels as $\frac{W}{300}$ at the points of contraflexure and as $\frac{W}{200}$ at the centre of the column. This additional shearing force is to be added to the shearing force accompanying the primary bending. See his remarks regarding the distribution of the shearing force along the member.

In view of the small magnitude of the shearing force in a column, the bracing designed to carry it should be made of the same size from one end of the column to the other. No economy is likely to be effected by varying the size of the lattice bars.

ARRANGEMENT OF BRACING.—Although the effect of the shearing force in a column is small, yet the proper arrangement of the web or lattice bracing to carry it is of great importance. There are many types of web bracing in general use. They may be divided into two classes: (i) lattice or other systems of cross bracing, (ii) batten plates.

Moore (Basquin, 1913) made some experiments on beams with different types of web bracing, including batten plates, the results of which are shown in Fig. 59. In this figure the efficiency of a solid web is assumed at 100 per cent., and the diagram represents the efficiency per cent. of the various types of braced webs. From these experiments it would appear that the double lattice

* If the initial deflection due to variations in the modulus of elasticity be taken into account, this figure will of course be increased.

system with a rivet at the crossing of the lattice bars is the best system, and the single system of lattice bars with independent ends the worst. Batten plates are not very much better. At low loads a properly arranged system of single lattice bracing is as good as a double system without rivets at the crossing of the bars, but the latter has the advantage at higher loads. It is doubtful, however, to what extent such experiments made on beams apply to columns.

The experiments made at Watertown Arsenal (1909–10) appear to be more valuable. In this case the specimens with single lattice bars and with double lattice bars (Exps. Nos. 1910 and 2084, Fig. 60) were equally strong, although the latter have a rivet where the bars cross. It should, perhaps, be remarked that the specimen with the double lattice bars deflected chiefly in a direction at right angles to the plane of the bracing. The strength of the specimen with

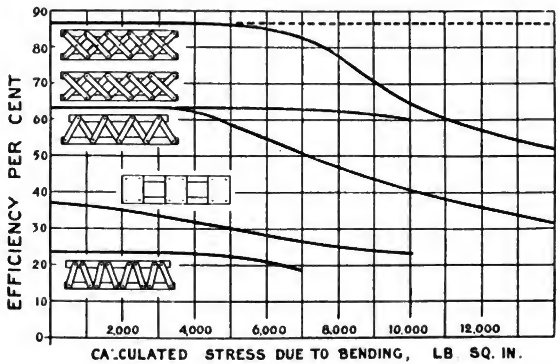



FIG. 59.

lattice bars with independent ends (Exp. 2085, Fig. 60) was very little inferior to that of the others, in spite of the increased length of the elementary flange columns. The addition of tie plates (Exp. No. 2083) did not increase the strength of this specimen. The broad flange beam (Exp. No. 2082) was the weakest of all.

So far as can be judged from these experiments, the double lattice system is the best form of web bracing, although it is probable that the single system such as is shown in Fig. 60, Exp. No. 1910, is equally as good where the shearing force is not too great. Alexander (1912) remarks that the double lattice system, Fig. 61 *(a)*, is the best of all. It is a mistake, however, to introduce a transverse member such as is shown in Fig. 61 *(b)*. If such a member be introduced, the web bracing at once partakes of the longitudinal load, for were it sufficiently strong it could obviously carry the whole longitudinal load without flanges at all. As has been seen (Howard, 1911), a lateral extension accompanies the longitudinal contraction, and if no transverse bar be introduced, the combination of lateral extension with longitudinal contraction leaves the length of the diagonal braces unaltered, and hence no stress occurs in them due

to the direct compression. If, however, a transverse bar be introduced, the lateral extension is prevented (see the effect of the diaphragm plate in Howard's experiments), and the longitudinal contraction must be taken up by the lattice bracing, which will thus be stressed independently of the shearing force. This was pointed out by Fidler (1887), and Alexander (1912) remarks that it is

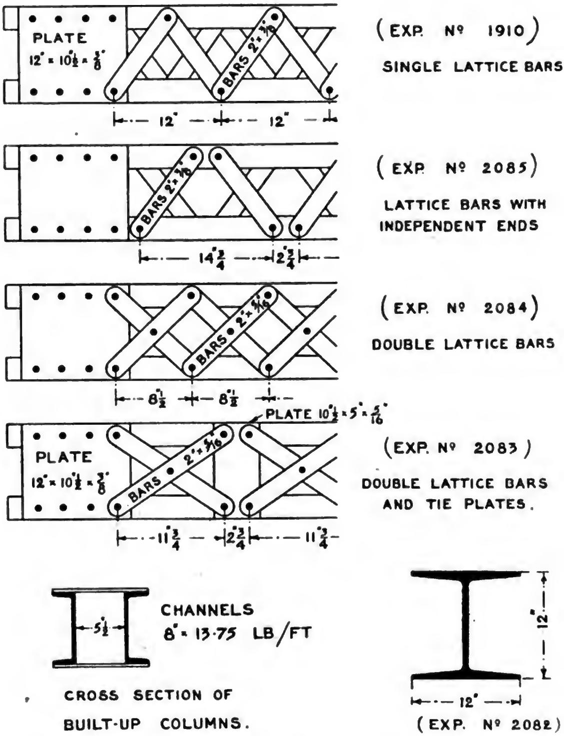



Fig. 60.—Types of Web Bracing (Watertown Arsenal, 1909–10).

surprising how great these stresses are. It seems probable, in fact, that the effect of a diaphragm plate may be quite other than is expected, for it may result in undesirable stresses in the web bracing. There seems room for some other arrangement for holding compression members in shape. It may be added that the member which failed in the Quebec Bridge had bracing of this undesirable type.

That lattice bracing is the best or most efficient form of web bracing in a column has been challenged by Emperger (1908), who made a number of experiments on built-up flat-ended columns with various types of web bracing (Fig. 62). He found, as might have been expected, that bracing consisting of single riveted batten plates (No. III) was utterly insufficient, for the flanges acted each as a separate column. Nevertheless, by the use of suitable and suitably spaced batten plates, it is possible to make a batten-plate column equally as strong as a similar lattice-braced member. Compare his experiments A IV and A VI with A V, and B V with B VI. The batten plates should be double riveted, and the ratio of $\frac{L}{\kappa}$ for the flange, considered as a column between the panel points, about one-half that of the column as a whole. On this account Emperger considers that lattice bracing is a useless expense, both in material and labour.

So far as Emperger's experiments go, therefore, it would appear that batten-plate columns may be made quite as efficient as lattice-braced columns, and they are obviously less expensive. This type of column has, in fact, been badly treated. It has been condemned as unscientific, contemptuously dismissed as "a bundle of faggots," improperly designed, treated as a solid column and by solid column formulæ, and where failure has occurred due to such methods of design the odium has been laid on the type of column instead of on the designer.

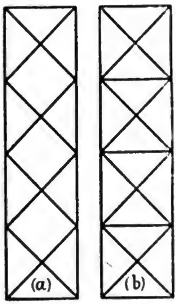


FIG. 61.

Now if Krohn's analysis be worth anything, it shows that so far as the strength of a built-up column is concerned, the consideration of first importance is the pitch of the panel points, and the necessities of the case in this particular can be quite as well attained in a batten plate as in a lattice-braced column. Since the shearing force in a column is small, it follows that no very serious forces or bending moments have to be carried by the batten plates and the riveting therein; they can therefore be economically arranged. Cases are, nevertheless, conceivable in which the pitch of the panel points must be so small that a lattice system would be more economical. That a batten-plate column is necessarily unscientific in design hardly needs refutation now that Vierendeel has shown that bridges of large span can be built economically without diagonals.

Theories for batten-plate columns have been given by many writers. That of Krohn (1908) is probably the simplest. Müller-Breslau's (1911) is the most exhaustive. Engesser's work in this connexion also should be mentioned (1909 and 1911).

In substance, the usual theory is as follows:—Let $F_s$ be the longitudinal shearing force on the group of rivets connecting a batten plate to a flange. Then $F_s \times h = Q \times j$, where Q is the actual shearing force on the batten plate, $j$ the pitch of the batten plates, and $h$ the distance apart of the flanges. Hence $F_s = \frac{Q \times j}{h}$. The bending moment on these rivets is $\frac{F_s \times h}{2} = \frac{Qj}{2}$. Therefore the rivets and the batten plates must withstand a longitudinal

shearing force $F_s = \frac{Qj}{h}$, and a bending moment $\frac{Qj}{2}$. As stated above, Krohn gives as the value of Q at the failure point $\frac{a}{28}$ metric tons, where $a$ is in

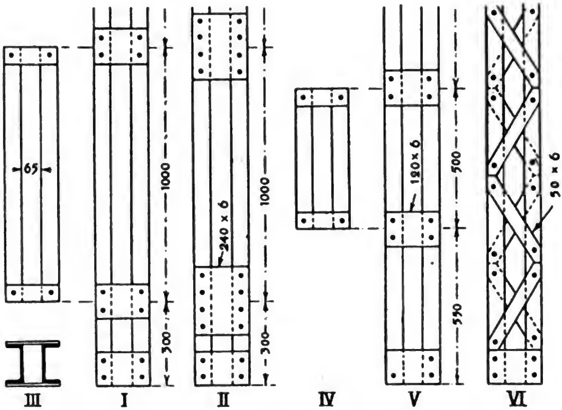



Fig. 62. Group B.

cm², or more generally $Q_{max} = \frac{a}{28}\frac{\omega}{\kappa}$ [equation (440)]. If preferred, however, Q may be given its value under working conditions as obtained on p. 194, namely 1 per cent. of the longitudinal load.

If the actual shearing force be carried by more than one line of batten plates, the longitudinal shearing force and bending moment found above must be shared between the lines.

## CHAPTER IV

**The $f_r$—$\frac{L}{\kappa}$ Diagram.**—The undeniable uncertainties which are inherent in any theoretical formulæ have led many to reject them altogether, and to resort entirely to empirical formulæ based on experimental failure loads.

Before examining these empirical formulæ, it will be well to consider the experimental results themselves, and the general characteristics which they display.

It is the universal practice at the present time to plot the experimental failure load per unit area ($f_r$) as an ordinate on an $\frac{L}{\kappa}$ base line. The resulting diagram, which may be termed the $f_r$—$\frac{L}{\kappa}$ diagram, exhibits the variation in the ultimate strength of the column with variation in the relative length.

Although there may be some justification for this method of plotting the experimental results, particularly for the greater length ratios (see some remarks of Considère, 1889), yet nothing is more certain than that $f_r$ depends on a great many other factors besides $\frac{L}{\kappa}$. The result is that the experimental results appear as a galaxy of points, " a milky way " as Emperger has happily expressed it, the shape of which depends largely on the material of which the specimens were composed. Figs. 63 (wrought iron), 73 (mild steel), 74 (cast iron), and 75 (yellow pine) are typical examples.

These diagrams well represent the variation in shape consequent on the difference in material, though the vertical depth of the area evidently depends on the magnitude of the imperfections in the conditions, for to these imperfections the reduction in strength is due.

In a ductile material like wrought iron or mild steel the upper limit of the area, which will evidently represent ideal conditions as nearly as is experimentally possible, is a curve the shape of which may be clearly seen in Fig. 63. The lower limit is a curve which corresponds roughly in shape to that obtained for eccentrically loaded specimens in which the eccentricity is considerable. The area enclosed is therefore of the form shown in Fig. 64.

The cast-iron diagrams are, on the other hand, somewhat different in shape (Figs. 72 and 74). Here the upper and lower limit curves are approximately of the same shape, and not very different in general form from the well-known Euler curve.

In the case of the timber specimens (Figs. 75 and 76) the variation in strength is so great that it is difficult to define the area within which the experimental points lie with any degree of exactitude. This large variation in the ultimate strength is undoubtedly chiefly due to large variations in the quality of the

material. On the whole the diagrams appear to approximate in shape to those of the wrought-iron and mild-steel specimens.

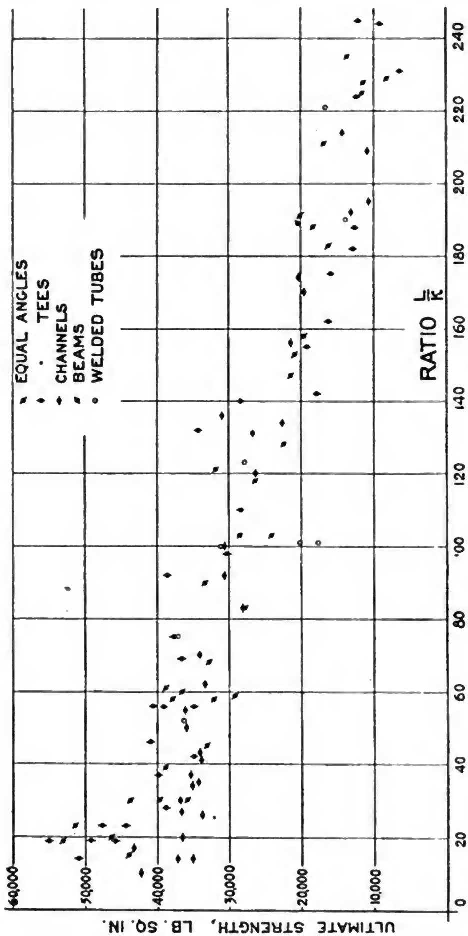



Fig. 63.—Experiments on Wrought-iron Specimens with Flat Ends (Christie, 1884).

The shape of the diagrams not only depends on the material, but also on the end conditions. Christie's average curves (Fig. 49) illustrate the effect of

different ends. Where the end conditions are very variable, the depth of the area is much increased. This is especially noticeable with hinged and flat ends.

One characteristic which all the areas have in common, however, is that when $\frac{L}{\kappa}$ exceeds a certain value the vertical width of the area is much decreased,

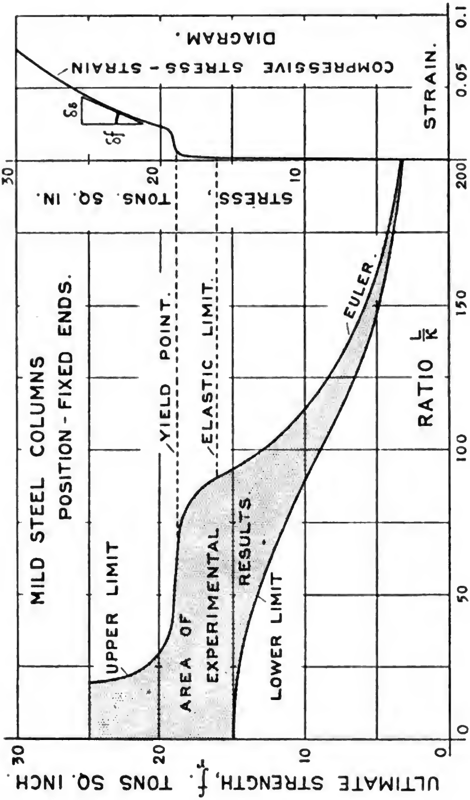



Fig. 64.

so that the experimental points may be said to lie on or near a single line. That line is the locus of Euler's formula, which for the reasons pointed out (p. 127) may be looked upon as giving the ultimate strength of long columns.

Below this limit of $\frac{L}{\kappa}$, since it is clearly impossible to represent an area by

a single line, any formula or connexion between $f_r$ and $\frac{L}{\kappa}$ must represent either the upper limit, the lower limit, or an approximate mean of the points.

The upper limit curve, of which the locus of Euler's formula may be considered to form a part, is evidently entirely independent of imperfections, since it represents the failure load under ideal conditions; and, therefore, for given end conditions, depends solely on the properties of the material. The shape of the lower limit and of the mean or average curves must obviously depend

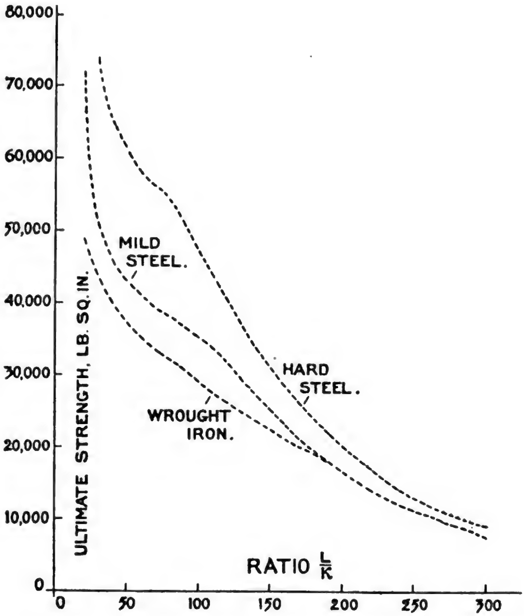



FIG. 65.—Average Strength Curves, Flat-ended Specimens (Christie, 1884).

on the magnitude of the imperfections. Some experimenters, by making the experimental conditions for all specimens as nearly as possible the same, and then averaging the results, have reduced the areas in question to lines. Others, again, have plotted merely the averages of all their experiments. The curves obtained in this manner, like the areas, exhibit certain peculiarities depending on the material and the end conditions. Christie's diagram of averages (Fig. 65) illustrates the difference in the ultimate strength of wrought-iron, mild-steel, and hard-steel specimens with flat ends. Lilly's diagrams (Fig. 66) also show the difference due to different materials. This author's remarks (1908) should

be consulted in conjunction with these curves. Howard's average curves (Figs. 48 and 50) and Kármán's figure (Fig. 67) are also cases in point.

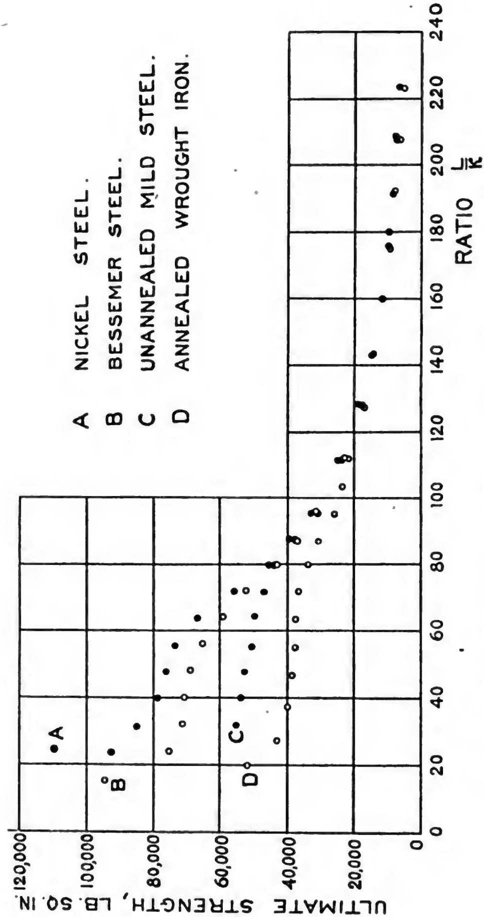



FIG. 66.—Ultimate Strength Curves, Solid Cylindrical Specimens (Lilly, 1908–10).

In general, all these curves present a generic resemblance. Not only so, but where care has been taken to straighten the specimens and to apply the load

concentrically, they resemble, in addition, the upper limit of the $f_r$—$\frac{L}{\kappa}$ area. Each curve may be divided into three portions corresponding roughly with the usual division of columns into long, medium, and short categories. That portion of the curve for the higher values of $\frac{L}{\kappa}$ corresponds, as has been seen, to Euler's formula. For the smaller values of $\frac{L}{\kappa}$ the curve rises very rapidly, showing great increase in $f_r$ as $\frac{L}{\kappa}$ diminishes. The connecting piece, corresponding to columns of medium length, is a more or less flat curve depending on the ductility of the material. In very ductile materials like wrought iron it is almost a horizontal line (Figs. 63 and 66), in brittle materials like cast iron or the hardest steels the three branches of the curve almost merge into one, or else the flat portion appears merely as a bump on the line (Fig. 66). One is reminded of the graphs of van der Waals's equation above the critical point. It is true that considerable differences exist in the shape of the curves of different experimenters, but the characteristics mentioned can be traced in them all.

The lower limit curve, on the contrary, presents none of these characteristics, except that it tends to merge into the Euler curve at large values of $\frac{L}{\kappa}$. It is, as has been mentioned, a curve like that obtained for eccentrically loaded specimens in which the eccentricity is considerable, and has the same general outline as the graph of the Rankine-Gordon formula.

**The Considère-Engesser Theory.**—That the upper limit curve depends solely on the properties of the material suggests that some connexion may exist between it and the stress-strain diagram. Lamarle (1846) proved that at the moment at which the long ideal column bends, the material passes the elastic limit, and concluded that for the shorter columns the upper limit was a straight line $f_r = f_e$. Considère (1889) suggested a direct connexion between the upper limit curve and the stress-strain diagram. He found that his experimental results fell away from Euler's curve at comparatively high values of $\frac{L}{\kappa}$, corresponding to loads of only from 9 to 10 kg/mm². His explanation of this was that, assuming the column to remain straight up to the point of fai ure and then to bend under the crippling load, if the proportional limit had been passed in direct compression, the ratio of stress to strain on the concave side of the specimen would no longer be equal to E, the tensile modulus, but would be something less. On the convex side, it is true, the modulus would still be equal to E, but the average modulus for the cross section as a whole would be less than E. This effect would be still more emphasized had the yield point been passed, for then $\frac{df}{ds}$ would fall very much in value. Hence for E in Euler's formula should be substituted a smaller modulus.

Considère does not seem to have carried his suggestion any further than this, but Engesser (1889) proposed a more definite connexion between the stress-strain diagram and the upper limit curve. He assumed, in effect, that the specimen remained straight up to the moment of failure, and that the

modulus of elasticity remained constant right across the cross section. As the load increased, the material would pass the elastic limit and yield point in turn; the modulus, or ratio of stress to strain $\frac{df}{ds}$, steadily diminishing. Nevertheless, Euler's formula would still hold if for E be substituted the diminished modulus $T_2 = \frac{df}{ds}$. The crippling load would therefore steadily diminish with the diminishing modulus, until at some point it would become equal to the increasing applied load, when the specimen would fail. Under these circumstances the modulus $T_2 = \frac{df}{ds}$ can, for a given value of the load, be at once obtained from the stress-strain diagram. Hence it is possible to predict from that diagram the shape of the $\left(f_r - \frac{L}{\kappa}\right)$ curve directly. Engesser later made some slight modifications to the above, in particular to the constants, with the object apparently of making the derived curve correspond more nearly to Tetmajer's straight line. In 1895, however, in reply to some criticisms by Jasinski (1895), he completely remodelled the whole theory. Jasinski, in what was practically a restatement of Considère's position, pointed out that when the slightest deflection occurs, the whole condition of affairs is altered. The modulus will no longer be uniform all over the cross section, for the deflection will reduce the stress on the convex side and increase it on the concave side. The material on the convex side is therefore in the condition of a specimen being unloaded, and the ratio of stress to strain will be equal to the modulus E, or nearly so. On the concave side, however, the modulus would vary right across the cross section; at any point, of course, being equal to the ratio of increase in stress to increase in strain corresponding to the particular stress at that point. Not only so, but the variation in the modulus would cause the position of the layer of fibres on which the stress due to bending is zero to move away from the central axis. In consequence, Engesser's analysis, in which, like Euler's, the modulus had been assumed constant all over the cross section, could not hold.

To this Engesser replied that the formula

$$W = \frac{\pi^2 ET}{L^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (450)$$

might, nevertheless, still be applied, although the value of T could not, of course, be calculated in the simple manner previously proposed. He gave an expression for T obtained on the assumption that the modulus is *constant* and equal to $\frac{df}{ds}$ on the concave side of the cross section, and constant and equal to E on the convex side, the dividing line being given by the condition $f_b = 0$. With this value of T, however, the shape of the upper limit curve is by no means easy to determine, and in his later writings Engesser finds T from an assumed straight-line variation of $f_r$ with $\frac{L}{\kappa}$, and not from the stress-strain diagram at all. This latter expedient was adopted by Schneider (1901), who

worked backward and determined T from a mean $\left(f_r - \frac{L}{\kappa}\right)$ curve. Whatever may or may not be the merit of the theory as an explanation of the shape of the upper limit curve, it appears wholly illegitimate to determine T from any mean curve, for the latter obviously includes the effect of imperfections in the conditions (initial deflections, eccentricities, etc.), which the formula $W = \frac{\pi^2 ET}{L^2}$ evidently does not take into account.

Engesser's second theory was adopted and developed by Kármán (1910). Kármán makes the same assumptions as Engesser regarding the stress distribution, determines the same expression for the modulus T, and finds its value for each point on the stress-strain diagram. He is thus able to obtain a $\left(f_r - \frac{L}{\kappa}\right)$ curve exhibiting a close measure of agreement with his experimental results (Fig. 67).

The modified Engesser theory was also given by Southwell (1912), who expresses his result somewhat differently. He finds a new length L′ such that

$$W = \frac{\pi^2 S}{(L')^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (451)$$

where S is the moment of stiffness of the cross section at the centre of the column. Southwell's theory is identical with Engesser's, although the shape of his formula is different, and the same assumptions are implied.

Chapman (1914) revived (with slight modification) the original Engesser (1889) theory, and used it to interpret Hodgkinson's experiments.

In the analyses examined so far, use has been made of the actual stress-strain diagram to find the value of $T_2$. This involves the preparation of complicated curves before the value of $f_r$ can be obtained. In order, therefore, to obtain a simple practical formula, several writers have proposed to replace the true stress-strain diagram by two straight lines, one representing the elastic and the other the plastic portion of that diagram. Thus Vierendeel (1906) puts

$$T = E$$

and

$$T = A - B\,(f_a)$$

for the elastic and plastic conditions respectively, and Stark (1907) makes similar assumptions.

To the above theory, suggested by Considère and developed by Engesser and others, it is possible to offer objections, more particularly to the mathematical treatment. Nevertheless, it appears probable that it has some basis in fact. However this may be, certain definite facts must be admitted.

In the first place, the shapes of all upper limit curves and, as has been seen, when the experimental conditions are good, not a few of the mean curves, have the same characteristics, which depend on the properties of the material. There is a definite connexion between the shape of the stress-strain diagram and that of the upper limit curve. Where, in materials like cast iron, the

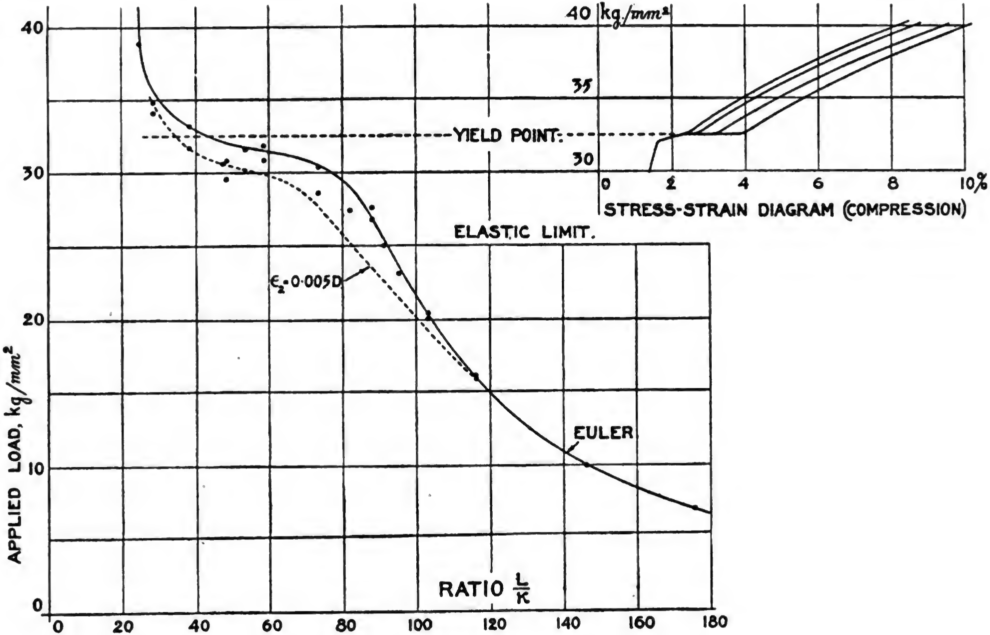



FIG. 67 (Kármán).

former is a smooth continuous curve, the upper limit curve is a smooth continuous curve (Fig. 74). Where, as in ductile materials like wrought iron, there is a marked yielding at the yield point, the upper limit curve exhibits a flattened shape to correspond (Figs. 63 and 66).

In the second place, it may be taken as proved that it is the elastic limit (or perhaps the yield point) of the material on which the compressive strength of a column of ductile material depends, and when this is overstepped in the extreme fibres, the column may be said to have failed. Lamarle (1846) showed that the moment a long ideal column bends, the elastic limit is exceeded and the column fails. He suggested that short columns fail when $W = f_e a$. Considère (1889) remarks that the resistance to crippling depends, above all, on the elastic limit.

Marshall (1887), as the result of his experiments, came to the conclusion that, so far as solid bars are concerned, the elastic limit is the chief factor in determining the ultimate resistance of columns of ordinary length made of wrought iron or mild steel, excepting the very hardest kinds; and that the elastic limit in compression and the ultimate compressive strengths are identical within a considerable range of length-ratio of columns. Judging from the figures given, the ultimate resistance is equal to the yield point when $\frac{L}{\kappa}$ is less than 100 for both flat and hinged ends. Above this limit there is a distinct falling off.

Buchanan (1907) made a number of experiments on full-sized bridge members, and some surprise was expressed at the low ultimate strength recorded. Nevertheless, his critics were agreed that the well-designed columns failed when the stress in the material exceeded the elastic limit or, rather, the yield point.

Jensen (1908), as the result of a detailed examination of Tetmajer's experiments, comes to the conclusion that $f_c$ at the failure point is equal to the yield-point stress in every case where $\frac{L}{\kappa}$ lies between 70 and 100.

Howard (1908) remarks that it may be regarded as axiomatic that the ultimate strength of iron and steel members of the usual engineering proportions is limited to the elastic limit of the material. The shape of the compression stress-strain diagram has also an influence on the ultimate resistance of a column. A *jog* in the curve at the elastic limit, the steel yielding under reduced stresses, might lead to the prompt failure of a compression member if the elastic limit be reached, even locally.

In 1909 he remarks again that the influence of the elastic limit in limiting the ultimate resistance of columns is shown by his experiments. For ordinary lengths of columns, $\frac{L}{\kappa}$ from 50 to 100, the ultimate strength falls within the elastic limit zone (Fig. 50). The probable reason why the early Phœnix columns of iron showed greater resistance than the later columns of steel is because the steel yields so decidedly at its elastic limit.

Again, in 1911, Howard says that it is believed that the minimum value of the elastic limit, as found in the component parts, chiefly modifies the ultimate resistance of the column, although it is probable that the shape of the stress-strain diagram immediately beyond the elastic limit has an important influence. Hence variations of 25 per cent. and over in the elastic limit, as

found in the plates and angles, would overshadow those considerations which find expression in empirical formulæ for columns, and take no account of such variations.*

Lilly (1908), commenting on his experiments with round-ended columns, says that for values of $\frac{L}{\kappa}$ less than 120 and greater than 40 the ductility and strength of the material at the elastic limit or near the yield point, and the percentage elongation, play an important part in determining the strength of columns. In a later part of his paper he suggests the hypothesis that the load producing failure will bend the column to its proof deflection.

In the experiments on nickel-steel specimens (1910 and 1914) it will be found that the ultimate strength is a little less than the tensile elastic limit of the material. The superiority of nickel-steel specimens over carbon-steel specimens due to its higher elastic limit is plainly evident in these and other tests.

Greger (1912) remarks that it is now established beyond doubt that the value of the buckling load depends on the yield point (compression limit).

It has been seen that the variation in the ultimate strength consequent upon the past history of the material can well be explained on the hypothesis that the ultimate strength depends upon the elastic limit.

In the third place, columns of ductile material should, according to the theory, fail in three different ways depending on their length ratio. Thus the failure of long columns ($f_r < f_e$) should depend on the modulus of elasticity E. The failure of medium columns ($f_y > f_r > f_e$) should depend on the elastic limit and yield point, and that of short columns ($f_r > f_y$) on the properties of the material beyond the yield point. There is experimental evidence that this is the case.

The division of columns into long, medium, and short was made by very early experimenters (see Hodgkinson, 1840), and recently Kármán (1910) has given much consideration to this aspect of the subject. Lilly's remarks (1908) on the failure of specimens of different materials should also be consulted. Robertson (1914) makes a similar division of the methods of failure of concentrically loaded position-fixed members.

Briefly, long columns in which $f_r < f_e$ obey Euler's law. Their strength depends on the value of E, and there is not very much difference between the various brands of wrought iron and steel. The deflection is elastic, and increases under, practically speaking, a uniform load (Fig. 41), usually gradually. When the load is removed, the specimen is found to be uninjured.

In medium columns in which $f_r$ lies between $f_e$ and $f_y$, the ultimate strength depends on the ductility and strength of the material at or near the yield point. The deflection begins earlier and increases very suddenly when the maximum load is reached. After removal of the load the deformation is permanent.

Short columns, in which $f_r > f_y$ fail by direct crushing and flowing of the material, the ultimate strength rises to high values. When the load reaches the yield point a period of instability occurs, but the specimen recovers and the load goes on increasing. The deflection after the yield point has been passed exhibits not a little of the character of elastic deflection. Unless the cross section be a solid bar, irregular crippling sets in, and in thin tubes and certain rolled sections failure is chiefly due to secondary flexure.

* Compare the more recent conclusions based on the *Amer. Soc. C. E. Experiments*, *Eng. News-Record*, N.Y., June 28, 1917, p. 639.

It may, therefore, be taken as established that :—

1. The shape of the upper limit curve has a direct connexion with the shape of the stress-strain diagram (see Fig. 64).

2. In columns of ductile material of ordinary proportions $\left(\frac{L}{\kappa} = \text{from } 40 \text{ to } 100 \text{ approximately, if the ends be position-fixed}\right)$ the elastic limit and yield point are the chief factors in determining the ultimate strength.

3. Columns may, in fact, be divided into three groups: (i) long columns of which the ultimate strength is chiefly determined by the modulus of elasticity, (ii) medium columns in which the elastic limit and yield point determine the ultimate strength, and (iii) short columns which fail by direct crushing and flow of the material.

The divisions between these groups are not very definite, and vary with different materials.

The question remains, how do the known facts agree with the Considère-Engesser theory? In the first place, it is evident that Euler's crippling load is the load at which long columns fail, and until $f_r$ exceeds $f_e$ Euler's curve represents the upper limit of strength, as the theory requires.

Secondly, the ultimate strengths of columns of ductile materials and of medium length fall between the elastic limit and the yield point when the experimental conditions are good; and both the theoretical and experimental upper limit curves show a quick tendency to rise for small values of $\frac{L}{\kappa}$.

Thirdly, as Kármán (1910) has done, it is possible to determine by the aid of the theory an upper limit curve agreeing closely with the experimental values. Southwell's curves (1912) show the same characteristics, though the measure of agreement is not so good.

Fourthly, there is the evidence already quoted of a more or less definite connexion between the ultimate strength of the specimen, as shown by the shape of the upper limit curve and the shape of the stress-strain diagram.*

On the other hand, Schneider (1901) points out that according to the theory Euler's formula should hold until the elastic limit is reached, whereas in Tetmajer's experiments, where the elastic limit was 2·6 t/cm², the experimental values began to fall away from the Euler curve at from 1·4 to 1·6 t/cm².

Curiously enough, as has been seen, Considère himself observed a similar falling off from the Euler curve at low values of the load in his experiments, and it was this falling off which led him to formulate his theory. He would explain the matter on the ground that the absolute limit of elasticity is very low.

The mathematical treatment of the theory, however, is worth a little further consideration. Imagine a perfectly straight, perfectly centered ideal column with position-fixed ends, which has remained perfectly straight until the elastic limit has been passed. Suppose that for some reason or other it begins to bend, and consider the stress on any cross section. It is evident that the bending moment set up will increase the strain on the concave side and diminish it on the convex. As assumed in the theory, therefore, the fibres on the convex side will be in the condition of a specimen being unloaded after overstraining, and those on the concave side in the condition of a specimen which has passed

* See also Robertson's conclusions (1915).

the elastic limit, and is being continuously loaded. The stress due to bending anywhere on the convex side will be, therefore,

$$f_b = T_1 s_b$$

and on the concave side $$f_b = \frac{df}{ds} \cdot s_b = T_2 s_b,$$

where $T_1$ and $T_2$ are the ratios of increase in stress to increase in strain under the conditions named.

Since, however, the strain and stress due to bending will vary from zero at the centre to a maximum on the outside, it follows that the strain in any cross section will vary from the centre to the outside, and hence the value of $T_2$ will vary in each layer of fibres. Not only so, but since the value of the bending moment will vary at each cross section, the strain due to bending will be different in every cross section, and therefore the value of $T_2$ will vary from end to end of the column. At the extreme end where the bending moment is zero, the modulus will be uniform right across the section; at the centre of the column where the bending moment is a maximum, the modulus $T_2$ will exhibit its maximum variation.

This variation will have two effects. In the first place, the value of the moment of stiffness S which is given by the expression

$$S = \int_{u_2}^{u_1} Eu^2 \cdot da \quad \ldots\ldots\ldots \quad \text{(see p. 22)}$$

will vary from one end of the column to the other.

Now it was proved in Part II, Case I, Variation 4, that while the column remains elastic, the moment of stiffness S might be assumed constant and equal to $E_a I$, even though the modulus of elasticity be not constant. But after the elastic limit has been passed, it does not appear that the variation in S will be small, particularly in columns where the yield point has been passed and $T_2$ drops to very low values.

In order to integrate the differential equation

$$\frac{d^2y}{dx^2} + \frac{M}{(1 - s_a) S} = 0 \quad \ldots\ldots\ldots \quad (452)$$

however, it is necessary to assume that $S (1 - s_a)$ is constant, and this has been done in all the analyses given, although it is obviously incorrect.

In the second place, the effect of the variation in the modulus will be to move the layer of fibres unaltered in length by the bending moment away from the centre of area of the cross section towards the convex side. The position of this layer of fibres is given by equation (7)

$$\int_{u_2}^{u_1} Eu \cdot da = 0.$$

It is evident that since the value of the modulus (now $T_2$) varies in each cross section, the position of this layer of fibres will vary too, and the surface on which the stress due to bending is zero will no longer be plane.

If, however, the column fail at the same load under which the first deflection appeared, it is evident that the centre of resistance will remain coincident with the centre of area, in spite of the variation in the moduli; for it is the

distribution of the *load* over the cross section which fixes the centre of resistance, which distribution in the case under consideration is uniform, and is unaffected by the deflection. Provided, therefore, that the column fail under the same load at which the first deflection takes place, it may be treated as an originally straight column, and moments may be taken about the centre of area.

In this case, if one assume, for the purpose of treating the matter analytically, as Engesser and his successors have done, that S $(1 - s_a)$ is constant, and that $T_2$ is constant over the area $a_2$ on the concave side of the cross section, then equation (7), which determines the layer of fibres unaltered in length by the bending moment, becomes

$$\int_0^{u_1} T_1 u \cdot da = \int_0^{u_2} T_2 u . \, da$$

or

$$T_1 \bar{u}_1 a_1 = T_2 \bar{u}_2 a_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (453)$$

where $a_1$ and $a_2$ are the areas of cross section on the convex and concave sides respectively of the layer of fibres unaltered in length by the bending moment, and $\bar{u}_1$ and $\bar{u}_2$ the distances of their centres of area from that layer.

The moment of stiffness

$$S = \int_{u_2}^{u_1} E u^2 \cdot da = T_1 \int_0^{u_1} u^2 \cdot da + T_2 \int_0^{u_2} u^2 \cdot da = T_1 I_1' + T_2 I_2' = TI,$$

where $I_1'$ and $I_2'$ are the moments of inertia of the areas $a_1$ and $a_2$ respectively about the layer of fibres unaltered in length by the bending moment. Then

$$T = \frac{T_1 I_1' + T_2 I_2'}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (454)$$

as given by Engesser (1895) and Kármán (1910).

The solution to the differential equation (452) evidently leads to the modification of Lamarle's formula

$$W = \frac{\pi^2 TI}{L^2} (1 - s_a),$$

or since $s_a$ is small compared with unity, to the modification of Euler's formula

$$W = \frac{\pi^2 TI}{L^2},$$

given by Engesser and his successors.

In practice, however, the column will bend under a smaller load than that actually producing failure. When the load is increased after bending, a new factor is introduced, for with the varying moduli the distribution of load over the cross section will vary, the *increase* in load not being uniformly spread over the cross section. The centre of resistance consequently will move away from the centre of area, though it will still not coincide with the layer of fibres unaltered in length by the bending moment.

Since the variation in the modulus will be different in each cross section, the amount of this movement will vary for each cross section, being

zero at the ends and a maximum at the centre. In other words, the line of resistance of the column will no longer coincide with the central axis, but will be a flat curve, and the effect of the variation in the moduli will be equivalent to an *initial curvature* of the column.

Under these conditions the maximum compressive stress at the centre of such a specimen is given by equation (87)

$$f_c = E_2 \left[ (v_2 + \epsilon_5 + \epsilon_6) \frac{W}{S} \left\{ \epsilon_2 \sec \frac{aL}{2} + \frac{8\epsilon_1}{a^2L^2} \left( \sec \frac{aL}{2} - 1 \right) \right\} + s_a \right]$$

on the assumption, of course, that $S(1 - s_a)$ is constant. In the present case $\epsilon_2$ and $\epsilon_6$ are both zero; and $\epsilon_5 = \epsilon_1$, the initial deflection of the line of resistance at the centre. $E_2 = T_2$, $a^2 = \dfrac{W}{S(1 - s_a)} \simeq \dfrac{W}{S} \simeq \dfrac{W}{TI}$ if $s_a$ be neglected in comparison with unity. The equation reduces, therefore, to

$$f_c = T_2 \left[ (v_2 + \epsilon_1) \frac{8\epsilon_1}{L^2} \left( \sec \frac{aL}{2} - 1 \right) + s_a \right] \quad . \quad . \quad . \quad . \quad (455)$$

This expression gives the maximum stress in the column under the conditions assumed. If the value of $\epsilon_1$ and $f_c$ could be obtained for the moment of failure, a value might be found for $a$ and hence for the failure load, but the solution is not easy. It is evident, however, that the stress becomes indefinitely great when $\sec \dfrac{aL}{2} = \infty$, i.e. when $\dfrac{aL}{2} = \dfrac{\pi}{2}$ or when $a^2 = \dfrac{\pi^2}{L^2} = \dfrac{W}{S(1 - s_a)}$.

Hence the value of the load W producing an indefinitely great stress in the column is

$$W = \frac{\pi^2 S}{L^2}(1 - s_a) \simeq \frac{\pi^2 TI}{L^2}(1 - s_a) \quad . \quad . \quad . \quad . \quad (456)$$

$$= \frac{\pi^2 TI}{L^2} \text{ approximately.}$$

This is the value of W obtained by Engesser and Kármán.

In short, this value of W should be looked on, not as the actual crippling load of the specimen, but as an upper limit to which the ultimate resistance can never quite attain. Just as Bauschinger defined Euler's limiting load P as the load which would cause the already existing deflection in a column to become indefinitely great, rather than the limiting load under which a long column would remain straight, so Engesser's crippling load may be defined as the load which would cause the curvature of the line of resistance set up by the variation in the moduli to become indefinitely great, rather than the limiting load under which the medium column would cripple.

The column must inevitably fail before this value of the load is reached, and in this sense Engesser's equation may be looked upon as the equation to the upper limit curve.

**The Experimental Behaviour of Columns.**—Before proceeding to a consideration of the various empirical formulæ which have been proposed to represent experimental results, it may be well to examine what occurs when a concentrically loaded, position-fixed column is tested. It will be assumed

that the experimental conditions are good and that the imperfections are small. In this case the specimen will remain very nearly straight during the early part of the experiment, and the stress will be, practically speaking, uniformly spread over the cross section. This corresponds to what Tetmajer (1896) calls the "equal stress condition," and he calculates that a factor of safety of from $4\frac{1}{2}$ to 6 is necessary to keep the material in this state.

Sufficiently sensitive instruments, as Bauschinger (1887) remarks, will, nevertheless, show that slight deflections exist even with the smallest loads; and the stress on the concave side will grow rather more quickly than that on the convex side. No very great departure from the equal stress condition will occur, however, until the load reaches from 0·9 to 0·95 of P, Euler's crippling load, when the stress on the concave side will begin to augment very quickly, and that on the convex side will reach a maximum, begin to diminish, and may even change sign.

All this is shown very clearly in Figs. 68 and 69, which represent the increase of stress in the extreme fibres of the cross section with increasing loads. These diagrams have been obtained as follows:

In a position-fixed column of symmetrical cross section the stress in the extreme fibres is given by the expression

$$f_c = f_a\left(1 \pm \frac{v_2 y_0}{\kappa^2}\right)$$

which, from equation (86), becomes

$$f_c = f_a\left[1 \pm \frac{v_2}{\kappa^2}\left\{\epsilon_2 \sec\frac{\pi}{2}\sqrt{\frac{W}{P}} + \frac{8P\epsilon_1}{\pi^2 W}\left(\sec\frac{\pi}{2}\sqrt{\frac{W}{P}} - 1\right)\right\}\right],$$

or, since

$$\frac{W}{P} = \frac{f_a}{\pi^2 E}\left(\frac{L}{\kappa}\right)^2,$$

$$f_c = f_a\left[1 \pm \frac{v_2}{\kappa^2}\left\{\epsilon_2 \sec\frac{L}{2\kappa}\sqrt{\frac{f_a}{E}} + \frac{8E\epsilon_1\kappa^2}{L^2}\left(\sec\frac{L}{2\kappa}\sqrt{\frac{f_a}{E}} - 1\right)\right\}\right]. \quad (457)$$

From this equation Figs. 68 and 69 have been plotted.

In Fig. 68 the stresses in the extreme fibres $f_c$ are plotted as ordinates on a base line representing the load per unit area $f_a = \frac{W}{a}$, in tons per square inch. It has been assumed for convenience that

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}03,$$

about one-half of the value estimated for $\frac{\epsilon_2 v_2}{\kappa^2}$ from Tetmajer's experiments on specimens with pointed ends by several writers.

Three cases are considered, $\frac{L}{\kappa} = 30$, $\frac{L}{\kappa} = 90$, and $\frac{L}{\kappa} = 200$, corresponding to the three classes of columns short, medium, and long. Horizontal lines represent the elastic limit and yield points. In the case of the short column

$\frac{L}{\kappa} = 30$, it will be seen that the material passes the elastic limit on both concave and convex sides in quick succession. The stress due to bending is small, and there can be no question of "unloading" stress on the convex side. For all practical purposes it is merely a case of direct compression, and the column may be said to fail by direct compression like any short specimen.

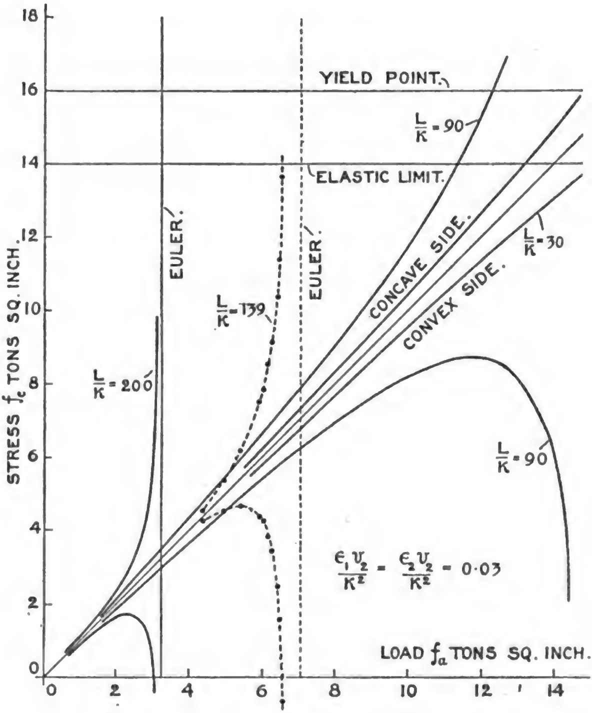



FIG. 68.

In the medium column $\frac{L}{\kappa} = 90$, the material passes the elastic limit on the concave side only. At that moment the stress on the convex side has about reached its maximum, but has not yet begun to decrease. It is evident that the column will have failed before the stress on the convex side becomes tensile. It may be said to fail by passing the elastic limit and yield point on the concave side.

In the long column $\frac{L}{\kappa} = 200$, it is evident that the material will never pass the elastic limit at all. The shape of the curve well indicates how great must be the stress due to bending and the corresponding deflection before the maximum stress reaches that limit. In the actual test the extreme deflection would throw the load off the specimen and it would remain bent under a reduced load; when removed from the machine it would be found uninjured. This, as is well known, is what actually happens. In this case the stress on the

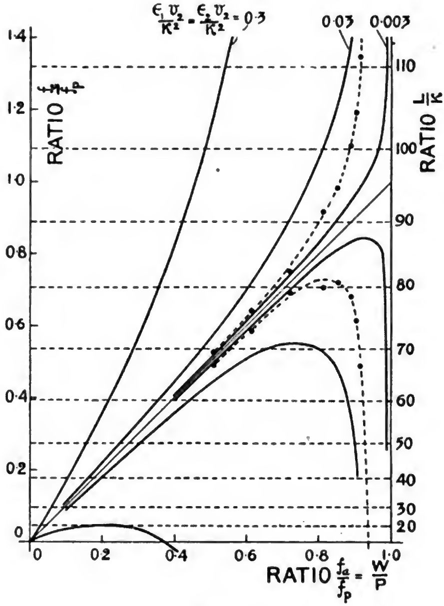



Fig. 69.

convex side may become tensile. The vertical asymptote represents Euler's crippling load for the specimen.

For comparison, the results of Tetmajer's Exp. No. 16, Table No. 4 (1896), $\frac{L}{\kappa} = 139$, have been shown on the same diagram.

In Fig. 69 the same equation is plotted somewhat differently.

Here the base line represents the ratio $\frac{W}{P}$ and the ordinates the ratio $\frac{f_c}{f_p}$.

Three cases are illustrated, viz.

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}003,\ 0{\cdot}03, \text{ and } 0{\cdot}30$$

respectively, the second of the three corresponding to the previous figure. In this diagram the elastic limit stress appears as a number of horizontal lines depending on the value of $\frac{L}{\kappa}$. The value assumed for the elastic limit is 14 tons sq. in. The results of Tetmajer's Exp. No. 16, Table No. 3 (1896), $\frac{L}{\kappa} = 166$, are also plotted (see the dotted curves).

From the diagram it is evident that for the case in which

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}003,$$

all specimens in which $\frac{L}{\kappa}$ is less than about 88 will pass the elastic limit in compression on both concave and convex sides. When

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}03,$$

the same is true for values of $\frac{L}{\kappa}$ less than about 72, and when

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}30,$$

the corresponding value of $\frac{L}{\kappa}$ is as low as 20.

The straight line at 45°, which corresponds to the ideal case, cuts the asymptote $\frac{f_a}{f_p} = 1$ at the point where $f_c = f_p$. If $f_c = f_p = f_e$, this corresponds to a value of $\frac{L}{\kappa}$ of about 96. This point is often called the limit of validity of Euler's formula. It is evident that it is the limiting value of $\frac{L}{\kappa}$ above which the stress on the convex side of the column will never pass the elastic limit in compression. Below this value of $\frac{L}{\kappa}$ the stress may or may not pass the elastic limit in compression on both sides of the column depending on the values of $\frac{\epsilon_1 v_2}{\kappa^2}$, $\frac{\epsilon_2 v_2}{\kappa^2}$, and $\frac{L}{\kappa}$.

Thus, for the cases in which

$$\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}03$$

for a value of $\frac{L}{\kappa} = 90$, the stress would never pass the elastic limit in compression on the convex side; but in the case of a column in which $\frac{L}{\kappa} = 60$, the stress would pass the elastic limit on both sides in fairly quick succession; and if the column remained elastic, the stress on the concave side would have reached a value more than double the elastic limit stress before the *compressive* stress on the convex side reached its maximum value. In an actual specimen, of course, the breakdown of the elasticity of the material would cause the bending stress on both sides to augment more quickly, which means that the maximum value of the compressive stress on the convex side would be reached more rapidly; but, on the other hand, the maximum stress on the concave side would increase very rapidly indeed.

A consideration of these diagrams leads in fact to the recognition of three different manners of failure:—

1. By elastic deformation as seen in long columns with a small eccentricity of loading, where the deflection increases very very rapidly with a small increase in the load.

2. By the material on the concave side passing the compressive elastic limit and yield point, while the compressive stress on the convex side remains within the elastic limit, and may even become tensile.

3. By the material on both sides passing the compressive elastic limit.

The manner in which any particular column will fail depends on the magnitude of $\epsilon_1$ and $\epsilon_2$ as well as on the value of $\frac{L}{\kappa}$. It does not correspond in any way with the division of columns into long, medium, and short depending solely on the value of $\frac{L}{\kappa}$.

Another point brought out by these diagrams is the fact that in straight, well-centered specimens, excepting those in which the value of $\frac{L}{\kappa}$ is large, no tension will exist in any of the fibres until after the column may be said to have failed. The common idea regarding the deformation of a column is somewhat erroneous. It is difficult to escape the ingrained impression, obtained from the figures used in the proof of column formulæ, of a considerably bent member with a tension and compression side, an idea confirmed by the usual photographs of specimens which have failed. These photographs, taken long after the maximum load has been passed and after the specimen has greatly deflected under a reduced load, give a totally wrong impression of the deflection which occurs in a column. In Fig. 70 will be found two diagrams, shown to a scale of five times full size, giving the actual movement of the centre point of two columns. In Exp. No. 1915 the specimen was 20 ft. 6¾ in. long, $\frac{L}{\kappa} = 150$, and had spherical ends. In Exp. No. 1947 the specimen was 6 ft. 5½ in. long, $\frac{L}{\kappa} = 47$, and had pin ends. The relative insignificance of the deflection is at once apparent.*

* See a remark by Talbot and Lord (1912) regarding the insignificance of the deflection in their experiments.

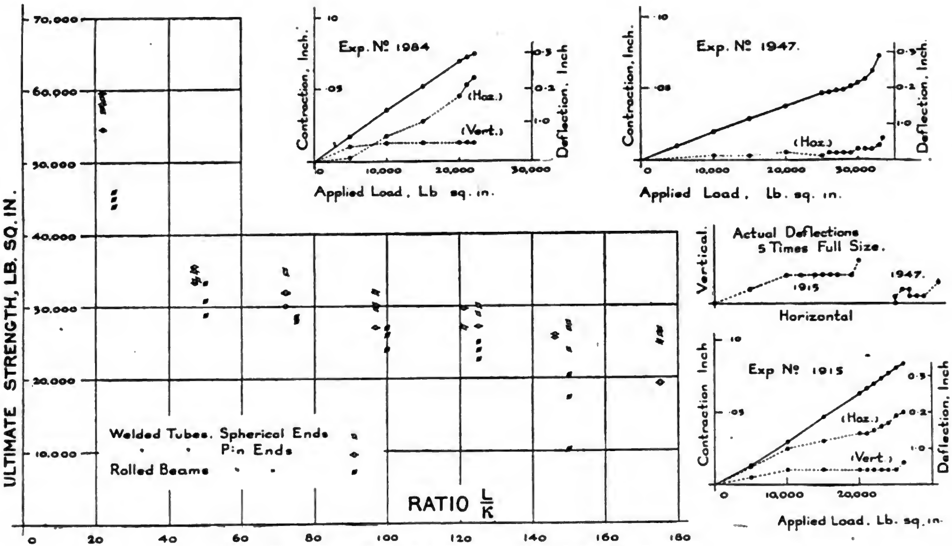



FIG. 70.—Watertown Arsenal Experiments (1909–10).

A suggestive experiment was made at Watertown Arsenal in 1908. A welded tube, $\frac{L}{\kappa} = 150$, was cut in half, the two halves simply rested the one on the other, and the whole tested as a column. The ultimate strength was rather more than the average of similar uncut specimens.

Love (1851) pointed out that in Hodgkinson's experiments "the deflection of the pillar produced by the breaking load never reaches half the diameter of the pillar," from which he argues that up to the point of maximum resistance no part of the cross section is ever in tension. This is hardly correct, but the fact remains that in columns of ordinary proportions, reasonably straight and well centered, the stress is always compressive.

The full line in Fig. 71 shows the actual distribution of stress over the cross section of a specimen in which $\frac{L}{\kappa} = 90$, and $\frac{\epsilon_1 v_2}{\kappa^2} = \frac{\epsilon_2 v_2}{\kappa^2} = 0{\cdot}03$, at the moment when $f_c = f_e$. It has been taken from Fig. 68, and is drawn to scale. The dotted line shows roughly the distribution of stress very soon after the yield point has been overstepped. This line shows clearly the result of a sudden yielding of the material, represented by a *jog* in the stress-strain diagram. It is evident that the flattening of the stress curve will have the effect of moving the line of resistance away from the central axis just as in the case of the ideal column previously discussed. In short, the moment the elastic limit is exceeded, a new eccentricity of loading, or rather an extra curvature of the line of resistance, will be set up, due to the variation in $\frac{df}{ds}$, which will continue to increase until the yield point is reached, when there will be a sudden jump in $\epsilon_1$ consequent on the *jog* in the stress-strain diagram. Here, then, is the probable explanation of Howard's remark (1908) that a *jog* in the stress-strain diagram at the elastic limit may lead to the prompt failure of a compression member.

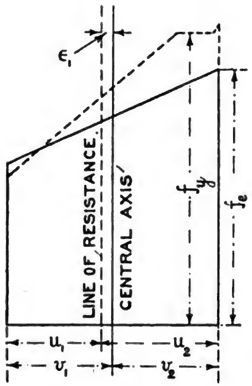



FIG. 71.

This increasing value of $\epsilon_1$ must evidently bring about the early failure of the specimen, for a sort of compound interest law will be at work. The increase in $\epsilon_1$ will increase the deflection, which in turn will cause a fresh increase in the value of $\epsilon_1$, and so a very rapid increase in the deflection even under a practically constant load must be expected. Tetmajer and others have recorded that columns of medium length fail very quickly when the ultimate strength is reached.

From this point of view it would seem that an ordinary column of medium length may be said to have failed when the stress in the extreme fibres has reached the yield point. This, as has been seen, is the conclusion of most investigators. It further follows that the eccentricity formula [equation (408)] should represent the ultimate strength of such specimens with a fair degree of

accuracy, provided that $f_c$ be put equal to $f_y$ and the correct values of $\epsilon_1$ and $\epsilon_2$ be inserted.

This formula will give the crippling load of long columns too, for it is evident from Fig. 68 that the value of W which makes $f_c = f_y$ $\left(\frac{L}{\kappa} = 200\right.$ and $\left.139, \text{ for example}\right)$ is very nearly equal to Euler's crippling load, or rather to a load a little less, as it should be.

It may be concluded, therefore, that the eccentricity formula can, without great error, be used to represent the ultimate strength of columns which fail in the first two of the three manners distinguished above. As a matter of fact, Marston (1898) showed that the formula would represent the average of Tetmajer's experiments on wrought-iron and mild-steel specimens with considerable accuracy, and Prichard pointed out the same thing in the case of Lilly's experiments (Lilly, 1913) on mild-steel specimens with round ends when $\frac{L}{\kappa} > 40$.

With columns which fail in the third manner, viz. by passing the elastic limit on both sides in quick succession, the condition of affairs is quite altered. This class includes all the shorter columns in which the eccentricity is small. Here the increased initial deflection due to the shift of the line of resistance is much smaller, and merely delays, without preventing, the material on the convex side passing the elastic limit. The variation in $\frac{df}{ds}$ is therefore much more nearly equalized, the column will continue to resist the load without undue deflection, flow of the material will be set up, and the ultimate resistance will rise to high values.

Under these circumstances the eccentricity formula cannot represent the experimental results, and, as is well known, it does not. Nevertheless, if the original eccentricity be large, a short column will fail in the second manner instead of the third (Fig. 69), in which case the eccentricity formula will still apply. Hence it follows that the formula will represent the lower limit of strength with reasonable accuracy, but not the upper limit when the value of $\frac{L}{\kappa}$ is small.

The above reasoning applies more particularly to columns of ductile material. In columns of brittle material such as cast iron, which possesses no marked elastic limit or yield point, there can be no marked distinction between long and medium columns, and the $f_r$—$\frac{L}{\kappa}$ diagram exhibits none. Short specimens fail by cones or wedges shearing out. Long specimens behave approximately as elastic specimens, and to them Euler's and the eccentricity formulæ may be applied.

**The Choice of an Empirical Formula.**—It will be recalled that the experimental failure loads, plotted on a $f_r$—$\frac{L}{\kappa}$ diagram, form scattered groups extending over a large area of varying width and shape. This area becomes

thin and narrow as the value of $\frac{L}{\kappa}$ gets larger, but spreads out in some cases to very great relative widths for the smaller values of $\frac{L}{\kappa}$. In some cases, Fig. 74 for example, the area is compact and easily defined; in others, Fig. 75, it is exceedingly diffused and irregular. Even leaving out very exceptional points, which may be looked on rather as danger signals or warnings to users of formulæ than normal experimental values, there remains in some cases an exceedingly wide area which is to be represented by a line, of which the equation is the desired empirical formula.

This is, of course, impossible, and it becomes necessary to decide whether the formula shall represent (i) the upper limit of the area, (ii) the mean or average of the experimental values, or (iii) the lower limit of the area. Regarding (i), the upper limit, there is little to be said from a practical point of view. The user of an empirical formula wants to know at what load the ordinary column will fail, and certainly an upper limit formula will not tell him that. Hence, whatever be the merits of the Considère-Engesser theory, it has little practical value. The question becomes, therefore, should the formula represent the mean or lower limit of the experiments? Much controversy has taken place on this point. In favour of the mean line it is urged that the factor of safety should cover the exceptional cases. As Tetmajer puts it, the lower limit line represents the exceptions rather than the rule. On the other hand, it is argued that the smallest load which may cause failure is, in fact, the strength of the column. Even lower limit formulæ, however, seldom include the very isolated cases.

Except for short columns, there is not much in the point after all. Where the lower limit curve is of the same shape as the mean curve, and the same working load is obtained by taking, say, one-quarter of the lower limit load or one-fifth of the mean or average load, it is evident the difference can be adjusted by the factor of safety. In the case of short columns, in which, as has been seen, failure may take place in two different manners, the lower limit has the advantage that it expresses the worst conditions; but even here the usual mean line formula neglects the upward rise at low values of $\frac{L}{\kappa}$. It is true that the ratio, ***mean load ÷ lower limit load***, varies with the value of $\frac{L}{\kappa}$, but within practical limits the variation is not large.

Provided a suitable factor of safety be chosen, therefore, either line may be adopted. Possibly the most suitable course would be to use the mean line, but make the factor of safety a function of the width of the area, i.e. of the ratio

$$\frac{\text{upper limit load} - \text{lower limit load}}{\text{mean load}},$$

the endeavour being to express in the formula the possibility of wide variation from the mean value of the load.

To determine the required formula it might appear that the obvious course is to determine the mean points and the equation to the curve on which they lie, or alternatively the equation to the lower limit curve. To this proceeding there are two objections. In the first place, it is very improbable that the constants obtained will bear any relation to the properties of the material, and

therefore the resulting equation will apply only to the particular brand of material employed in the experiments. In the second place, the result is a complicated formula unsuitable for practical use.

What is required is a simple expression of general applicability. To obtain the desired end two methods have been adopted. Either a theoretical formula has been taken and, by the introduction of constants or by variation of those already existing, converted into an empirical formula; or a simple equation (usually that of a conic section) has been adopted, and the constants determined to suit as nearly as possible the experimental results. By the first expedient factors representing the properties of the material are introduced into the equation which render it adaptable to different materials. By the second, a simple formula is obtained, the constants of which may or may not have any relation to the properties of the material. Of the first type, Euler's, the eccentricity, and the Rankine-Gordon are the most important. Of the second type, which includes every kind of conic section, logarithmic and other curves, the most important are the various straight lines and the parabola. Equations which represent only a few experiments of a particular type, or their proposer's particular views, need hardly be considered.

Whatever be the origin of the formula, it must represent the average, or lower limit if preferred, of the experimental results. It should be simple, easy to apply, and adaptable, that is to say, the constants should bear some definite relation to the properties of the material. It remains to be seen to what extent the formulæ commonly proposed fulfil these conditions.

**Empirical Formulæ.**—The first empirical rules were those given by Musschenbroek (1729), who concluded that

$$R = c \cdot \frac{BD^2}{L^2}.$$

Girard (1798) and Gauthey (1813) also gave formulæ to represent their experimental results. Rondelet's table (1812) results in a curve not unlike the graph of the Rankine-Gordon formula. These early empirical rules, like those of Belanger (1858) and Bourdais (1859), are merely of historical interest.

Hodgkinson (1840) proposed the formula

$$R = c \cdot \frac{D^n}{L^m},$$

analogous to Euler's formula, to represent his experimental results, but it has been generally recognized as very inconvenient. Hodgkinson's experiments, however, led to the first practical formula for the ultimate strength of columns. To represent these experiments, Gordon revived Tredgold's theoretical formula in the form

$$R = \frac{c_1 a}{1 + c_2 \left(\frac{L}{D}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (458)$$

and determined the constants from Hodgkinson's results. Hodgkinson himself (1840) gave a formula of essentially the same type.

Love (1851), who objected to both Tredgold's and Hodgkinson's formulæ on the ground that neither introduces the crushing strength of the material,

plotted as ordinates $\frac{R}{f_c a}$ from Hodgkinson's experiments, on a $\frac{L}{D}$ base line. He assumed as the equation to the mean curve

$$y = c_1 x^m + c_2,$$

and obtained a formula

$$R = \frac{f_c a}{c_1 + c_2 \left(\frac{L}{D}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (459)$$

which, though entirely empirical, is of the same variety as Gordon's. He gives a similar formula for the results of his own experiments on steel specimens.

Schwarz (1854), Laissle and Schübler (1857) in Germany, and Rankine (1866) in Great Britain, proposed a formula similar to Gordon's, but in which R is a function of $\frac{L}{\kappa}$ instead of $\frac{L}{D}$.

$$R = \frac{c_1 a}{1 + c_2 \left(\frac{L}{\kappa}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (460)$$

This is commonly known as the Rankine-Gordon or Schwarz-Rankine formula. It is essentially a theoretical formula, but is most frequently employed empirically to represent the results of experiments. Many and various have been the constants proposed for this formula.

It has been seen that the shape of the lower limit curve and, except for small values of $\frac{L}{\kappa}$, the shape of the average curve bear a considerable resemblance to the shape of the graph of the Rankine-Gordon formula; and hence it would appear that this formula, with suitable constants, should represent experimental results. To what extent this is true in the case of Hodgkinson's experiments on cast-iron specimens may be judged from Fig. 72. Winkler (1878) compared both the Gordon and the Rankine-Gordon formulæ with the results of the Cincinnati Southern Railway experiments. He came to the conclusion that both the formulæ gave correct results, and that the constant $c_1$ was approximately proportional but not equal to the tensile strength of the iron, and differs in the two formulæ. Bouscaren (1880) calculates the values of the constant $c_1$ in the two formulæ from the same experiments, and comes to the conclusion that the Rankine-Gordon is the more correct of the two.

Clarke, Reeves and Co., as the result of their experiments on Phœnix columns (1882), conclude that Gordon's formula does not express the true strength of these columns, and suggest that two formulæ are required—one for values of $\frac{L}{D}$ less than 15, the other the values of $\frac{L}{D}$ greater than 15. Nevertheless, Bouscaren shows that for lengths greater than 10 diameters a Rankine-Gordon formula will represent the results with considerable accuracy.

The constants in the formula should, however, be determined from the experiments which it is desired to represent, and not from reasonings based on the assumption that the formula has a rational basis. Cooper, in the

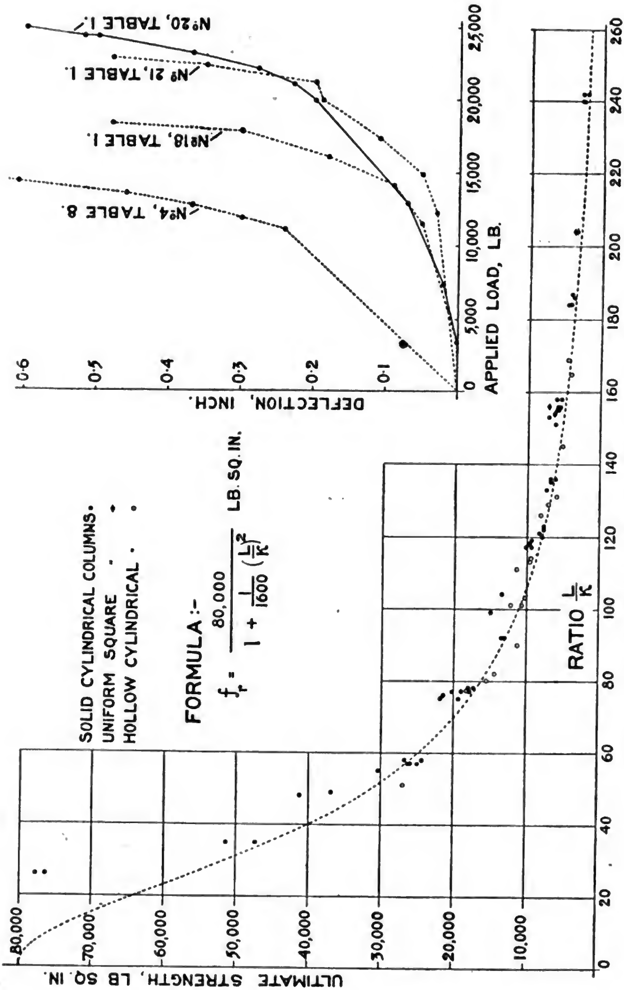



FIG. 72.—Cast-iron Specimens, Round Ends (Hodgkinson, 1840).

discussion on the paper by Clarke, Reeves and Co., insists that $c_1$ is merely a numerical constant taking the place of the factor $f_c$ in the theoretical formula, and has no relation to the crushing strength of a short column. Rankine, like Gordon, appears to have determined his constants from Hodgkinson's results, and the formula with these constants represents a low average of the experiments. On the other hand, the constants usual in the Schwarz and Laissle and Schübler formulæ give in some cases too high a value for the ultimate strength. Thus Bauschinger (1882) finds that the constant $c_2$ in the denominator should be 0·0006 for cast-iron columns in which the core is eccentric, and that the value 0·00022 only applies to perfect castings. See also his remarks in 1887 regarding the values of the constants for his wrought-iron specimens with flat ends. Gérard's article (1907) on the exactitude of the Rankine-Gordon formula may also be consulted.

The constants usually accepted in Great Britain are :—

| | $c_1$ lb. sq. in. | $\frac{1}{c_2}$ |
|---|---|---|
| Cast iron . . . . . . . . . . | 80,000 | 1,600 |
| Wrought iron . . . . . . . | 36,000 | 9,000 |
| Mild steel . . . . . . . . . . | 48,000 | 7,500 |
| Dry timber (strong varieties) . . | 7,200 | 750 |

These apply to columns with round ends. For other end conditions use $q$L in place of L.

Lilly (1908 and 1910) has given a fresh set of constants for the Rankine-Gordon formula based on his own experiments. The conditions in these experiments appear to have been very good; in fact, Lilly's curves for concentrically loaded specimens, which are average curves, have the shape of upper limit curves rather than average or lower limit curves (Fig. 66). Nevertheless, the Rankine-Gordon formula is used to represent the results, though the accord is not very good, with the result that the values of the constant $c_1$ are much higher than is usual. This should be borne in mind when using these constants.

Seaman (1912) has plotted the results of very many experiments on one diagram, and has determined constants for an approximate average curve.

Emperger (1897) supposes that the Rankine curve becomes approximately tangent to the Euler curve at the validity limit of the latter. His constants for the former then are $c_1 = f_y$, $c_2 = \frac{0{\cdot}6\, f_y}{c_3 E}$, which depend directly on the properties of the material. As a lower limit curve for Christie's, Tetmajer's, and other experiments on wrought-iron specimens* with round and pointed ends, he suggests the constants $c_1 = 2{\cdot}6$ t/cm² and $c_2 = 0{\cdot}0001$; and for flat ends $c_1 = f_y = 2{\cdot}6$ t/cm² and $c_2 = \frac{0{\cdot}6\, f_y}{c_3 E} = 0{\cdot}000032$. To allow for practical inaccuracies he would increase $c_2$ to 0·00005. It may be of interest to point out that actually the Rankine-Gordon formula is tangent to the Euler curve when $\frac{L}{\kappa}$ = infinity (Merriman, 1894), in which case $c_2 = \frac{f_c}{c_3 E}$.

* For mild-steel specimens $c_1 = 3{\cdot}1$ t/cm² (1907).

In addition to the ordinary Rankine-Gordon formula, the eccentricity form of that equation

$$R = \frac{c_1 a}{1 + \frac{\epsilon_2 v_2}{\kappa^2} + c_2 \left(\frac{L}{\kappa}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (461)$$

has been used to represent experimental results (see, for example, Pullen, 1896). De Préaudeau (1894) proposes the modification

$$f_r = c + \frac{c_1}{1 + c_2 \left(\frac{L}{\kappa}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (462)$$

where $c$ is a constant which takes into account variations in the properties of the material. Bender (1885) would write the formula

$$R = \frac{c_1 a}{1 + c_2 \left(\frac{L}{\kappa}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (463)$$

and Bredt (1894).

$$R = \frac{c_1 a}{1 + c_2 \left(\frac{L}{\kappa}\right)^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (464)$$

which he claims agrees better with the eccentricity formula than the ordinary Schwarz-Rankine formula.

It has been seen that the eccentricity formula (408)

$$W = \frac{f_c a}{1 + \frac{v_2 \epsilon_2}{\kappa^2} \sec \frac{L}{2}\sqrt{\frac{W}{EI}}}$$

theoretically, at least, should represent the lower limit of column strength, and, except for small values of $\frac{L}{\kappa}$, the average strength, with a considerable degree of accuracy.

Both Fidler's (1886) and Moncrieff's (1901) formulæ are variants of the eccentricity equation (408), and a reasonably good agreement with experimental failure loads is obtained with these formulæ.

Nevertheless, neither the Rankine-Gordon nor the eccentricity formulæ are very convenient for practical application, and most of the constants proposed bear no obvious relationship to the properties of the material. These formulæ, therefore, appear to offer no advantages over the simpler equations which form the second type of empirical formulæ, provided the latter represent the experimental results with equal success. Of this second type of equation the straight line is the most common:

$$f_r = c_1 - c_2 \left(\frac{L}{\kappa}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (465)$$

This formula was first proposed by Burr (1882) to represent the results of Clarke, Reeves and Co.'s experiments. Emery and Merriman also proposed

straight lines, but made $f_r$ a function of $\frac{L}{D}$. As in the case of the Rankine-Gordon formula, the constants proposed have been many and various. Of these the most important are the Johnson (1886) and Tetmajer (1890 and 1896) constants. Strobel's (1888), to represent the results of his experiments on Zed-bar columns; Considère's (1889); Jasinski's (1894), to represent Considère's and Tetmajer's experiments; Müller-Breslau's (1911), to represent Kármán's experiments; and Hutt's (1912), may also be mentioned.

T. H. Johnson plotted the results of numerous experiments in the usual way, and came to the conclusion that, for the lower length ratios, the average curve is a straight line tangent to Euler's curve, and intersecting the vertical axis at a point which is constant for all varieties of end conditions. The point of tangency is the validity limit for Euler's formula, which represents the strength of the longer columns. Actually the average curve is a mean between upper and lower limit curves of the same type. A table is given which suggests that $c_1$ is equal to the modulus of rupture obtained by bending experiments, and from the condition of tangency $c_2$ is evidently a function of $c_1$ and E.

Tetmajer also proposed to use Euler's formula for the larger values of $\frac{L}{\kappa}$ and a straight line for the smaller, but his straight line is not tangent to Euler's curve (Fig. 73). His formula for mild-steel specimens,

$$f_r = 3{\cdot}1 - 0{\cdot}0114\frac{L}{\kappa} \text{ t/cm}^2 \quad . \quad . \quad . \quad . \quad . \quad (466)$$

is that most commonly used by the continental nations of Europe. It may be said to represent the average of his experimental results, although as Emperger and others have pointed out, his polygon of group-means exhibits rather the reversed curvature of the Rankine-Gordon and eccentricity formulæ. It is of interest to compare Jasinski's estimate of the constants, determined by the method of least squares from both Considère's and Tetmajer's experiments. He gives for mild-steel specimens

$$f_r = 3{\cdot}387 - 0{\cdot}01483\frac{L}{\kappa} \text{ t/cm}^2 \quad . \quad . \quad . \quad . \quad (467)$$

To Tetmajer's formula Emperger has raised the objection that the constants bear no relationship to the properties of the material. Müller-Breslau (1911) points out that the validity limit $\frac{L}{\kappa} = 105$ for Euler's formula does not correspond to the elastic limit $f_e = 2{\cdot}4$ t/cm² of the material, but to a considerably smaller value, $f_r = 1{\cdot}90$ t/cm², and also that for small values of $\frac{L}{\kappa}$ the formula does not represent the crushing strength of the material. He suggests the formula

$$f_r = f_e - f\left(\pi\sqrt{\frac{E}{f_e}} - \frac{L}{\kappa}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (468)$$

$\pi\sqrt{\frac{E}{f_e}}$ is the value of $\frac{L}{\kappa}$ at the validity limit of Euler's formula where $f_r = f_e$, and $f$ is to be obtained from the experiments in which $f_r \overline{\gtrless} f_e$. He remarks that

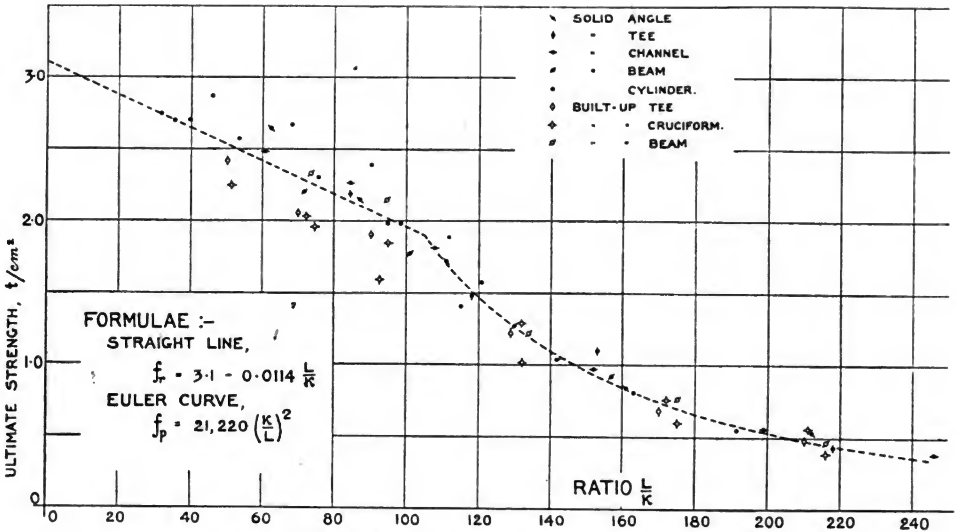



FIG. 73.—Mild-steel Specimens, Pointed Ends (Tetmajer, 1896).

the results of Kármán's experiments can be represented by this straight line quite as well as by Kármán's curve. He remarks also that in practice the eccentricity would be much greater than in Tetmajer's experiments.

There has been a tendency in many quarters to assume that the straight line formula

$$f_r = c_1 - c_2\left(\frac{L}{\kappa}\right)$$

is strictly analogous to the formula

$$f_a = f_c - f_b.$$

That is, to assume that $c_1$ is the maximum stress in the material at the point of failure, and $c_2\left(\frac{L}{\kappa}\right)$ the stress due to bending. This gives to the straight line formula a rational basis which it certainly does not possess.* Jensen (1908) has shown that the assumption that $c_1$ is the maximum stress in the material leads to absurd results.

The straight line should, in fact, be looked upon merely as an empirical expression representing the experimental results obtained for columns of medium length. It can claim no rational basis, and its constants bear, in general, no relationship to the properties of the material. Its one merit is simplicity.

In addition to the straight line, every other conic section has been proposed to represent experimental results. The only one of importance is the parabola.

In 1893 J. B. Johnson proposed to represent the ultimate strengths by a "parabola having its vertex at the elastic limit on the axis of loads, and tangent to Euler's curve." By "elastic limit" Johnson means the yield point, not the proportional limit. The equation to this curve is

$$f_r = f_y - c_2\left(\frac{L}{\kappa}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (469)$$

where

$$c_2 = \frac{f_y^2}{4c_3E}.$$

Johnson's values for the constants are given on pp. 231, 234, and 239.

The ultimate strengths for the higher values of $\frac{L}{\kappa}$ are, of course, to be represented by Euler's formula. The result of the combination of these two formulæ is a curve of the Rankine-Gordon or eccentricity formula type, composed of two parts, the parabola and the graph of Euler's formula, each of which has a simple equation. The constants have a definite connexion with the properties of the material, and the curves represent the average of the experimental results $\left(\text{except, perhaps, for low values of } \frac{L}{\kappa}\right)$ quite as well as any other equation. The validity limit of Euler's formula is such that, as a rule

* Talbot and Lord (1912) have shown, however, that $c_2$ varies at different periods in the experiment, increasing as W increases, which suggests that $c_2$ may be some function of $f_b$.

in practice, only the parabola need be used. A further advantage of the parabola is that it is rather easier to find the required dimensions by its use than by any other empirical formula.

To choose between the numerous formulæ proposed is not an easy task. Each formula has its own advantages and disadvantages, and may represent the results of certain experiments better than any other. Fortunately, the choice has been simplified, at any rate for ductile materials, by the work of Ostenfeld (1898). From the better-known experimental work this writer has determined the values of the constants for the more important empirical formulæ, and the corresponding mean error, by the method of least squares. In this way he has been able to determine the degree of exactness with which each represents the experimental results. He applied

1. The straight line—Euler formula,
2. The Johnson parabola—Euler formula,
3. The Rankine-Gordon formula,
4. A modification of the eccentricity formula,

to various experiments on concentrically loaded specimens. His conclusion is that formulæ 1 and 2 are better than 4, whilst 3 (the Rankine-Gordon formula), although not wholly inapplicable, gives the largest mean error. The best straight line is not a tangent to Euler's formula, so that T. H. Johnson's straight line is not so good as others. The tangent parabola (J. B. Johnson's) is very nearly the best possible. Both the straight line and the parabola give very serviceable results, but in most cases, and especially for the longer series of experiments, the Johnson parabola is somewhat more accurate than the straight line.

It would appear, therefore, that for ductile materials the Johnson parabola—Euler formula possesses the following advantages :—

It represents rather more accurately than any other the average ultimate strength of concentrically loaded specimens.

The two equations are simple and easily applied.

The constants have a definite connexion with the properties of the material.

The validity limit of Euler's equation is high, so that in practice only the parabola need be used.

It lends itself to an easy determination of the necessary area of cross section.

These advantages appear sufficient to warrant the preference being given to this formula; it remains to be seen to what extent the character of the material may modify this conclusion or affect the values of the constants.

CARBON STEEL.—For ordinary mild-steel specimens (percentage of carbon from 0·10 to 0·15), Johnson gives the formulæ :—

*Hinged ends:*

$$f_r = 42{,}000 - 0{\cdot}97\left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in., } c_3 = 16,\ X_p = 150 \quad . \quad . \quad (470)$$

*Flat ends:*

$$f_r = 42{,}000 - 0{\cdot}62\left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in., } c_3 = 25,\ X_p = 190 \quad . \quad . \quad (471)$$

To represent Tetmajer's results for specimens with pointed ends (percentage of carbon 0·08 to 0·12), Ostenfeld determined the formula

$$f_r = 2{,}724 - 0{\cdot}0874\left(\frac{L}{\kappa}\right)^2 \text{ kg/cm}^2,\ X_p = 125$$

$$= 38{,}740 - 1{\cdot}243\left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in.} \qquad (472)$$

For Christie's specimens with flat ends (percentage of carbon 0·11 to 0·15), Ostenfeld gives

$$f_r = 3{,}145 - 0{\cdot}061\left(\frac{L}{\kappa}\right)^2 \text{ kg/cm}^2,\ X_p = 162$$

$$= 44{,}730 - 0{\cdot}87\left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in.} \qquad (473)$$

If the value of $f_y$ be taken in round figures at 40,000 lb. sq. in., the formula becomes

$$f_r = 40{,}000 - \frac{4}{3}\left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in.} \qquad (474)$$

a simple and convenient form.* $X_p = 122$, $E = 30{,}000{,}000$ lb. sq. in.

For Christie's experiments on hard-steel specimens (percentage of carbon 0·36) with flat ends, Ostenfeld gives

$$f_r = 4{,}549 - 0{\cdot}117\left(\frac{L}{\kappa}\right)^2 \text{ kg/cm}^2,\ X_p = 140$$

$$= 64{,}700 - 1{\cdot}66\left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in.} \qquad (475)$$

He gives also modified parabolic formulæ to represent Considère's results.

Since it has been shown that the ultimate strength of columns of medium length depends chiefly on the yield point, it would appear that considerable advantage might be obtained by using a harder steel for compression members than for ties. It would probably pay manufacturers to roll a special steel merely for columns, with a high elastic limit and yield point, even if the plastic range were small.

Nevertheless, the advantages of a high carbon steel are considerably reduced in that the material suffers so much from shop treatment. Cold straightening and punching are highly injurious.

NICKEL STEEL.—In late years considerable attention has been given to the employment of nickel steel for columns. In addition to its high yield point, nickel steel possesses the important quality that it suffers very much less than the high carbon steels which approach it in strength from the unavoidable damage due to shop processes. Considerable advantage appears, therefore,

* Johnson gives this formula ($q = 1$) to represent the results of Tetmajer's experiments on mild-steel specimens.

to result from its employment for columns in which the ratio of $\frac{L}{\kappa}$ is not large. For long columns, where the ultimate strength depends chiefly on the modulus of elasticity, nickel steel has little advantage over mild carbon steel, for the modulus of elasticity is about the same for each. As examples of the superiority of nickel-steel columns, the following experiments may be quoted.

Waddell (1909) tested six carbon and six nickel-steel columns of similar design

*Carbon steel* $f_T = 65{,}000$ lb. sq. in.
$f_e = 35{,}000$ lb. sq. in.

*Nickel steel* $f_T = 100{,}000$ to $115{,}000$ lb. sq. in.
$f_e = 60{,}000$ lb. sq. in.
$3\frac{1}{2}$ per cent. nickel.

*Ratio of ultimate strengths* :—

$$\frac{L}{\kappa} = 27 \quad \frac{\text{Carbon}}{\text{Nickel}} = \frac{39{,}200}{68{,}700},$$

$$\frac{L}{\kappa} = 81 \quad \frac{\text{Carbon}}{\text{Nickel}} = \frac{30{,}500}{44{,}700}.$$

In Lilly's experiments (1910) the nickel-steel columns have a decided advantage over those of mild steel (Fig. 66). The percentage of nickel was 3.

Bohny (1911) tested four pairs of specimens, each pair consisting of two identical specimens, one of mild steel, the other of 2 to $2\frac{1}{2}$ per cent. nickel steel. The nickel steel had an ultimate tensile strength $f_T = 56$ to 65 kg/mm$^2$, and a yield point of not less than 35 kg/mm$^2$. The experiments showed that the nickel-steel specimens were nearly 50 per cent. stronger than those of mild steel.

The nickel-steel model compression members of the new Quebec Bridge (1910) were constructed of a 3·66 per cent. nickel steel, of which the ultimate tensile strength $f_T = 76{,}520$ to $91{,}300$ lb. sq. in. and the elastic limit varied from 53,590 to 68,360 lb. sq. in. They failed when the load reached a value a little less than this tensile elastic limit.

Hodge's specimens (1913), composed of a $3\frac{1}{4}$ per cent. nickel steel with an average elastic limit of 55,500 lb. sq. in., failed at a load somewhat greater than this figure.

The advantage of nickel-steel specimens may also be observed in the Watertown Arsenal experiments (1911).

To determine constants for a formula for nickel-steel specimens with the experimental data available is a little presumptuous. The quality of the material is very variable; not only do different brands of nickel steel differ widely, but there are considerable differences in the qualities of specimens of the same brand.

To represent his tests on round-ended specimens, Lilly (1910) proposes the Rankine-Gordon formula

$$f_r = \frac{120{,}000}{1 + \frac{1}{2{,}800}\left(\frac{L}{\kappa}\right)^2} \text{ lb. sq. in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (476)$$

This formula, like others proposed by Lilly, represents a high average of experiments in which the conditions were very good. Tensile tests of the material gave the values $f_T = 109{,}000$ lb. sq. in., $f_y = 80{,}000$ lb. sq. in., $E = 31{,}000{,}000$ lb. sq. in. The percentage of nickel = 3.

To represent average values for all nickel-steel specimens in which the percentage of nickel does not exceed 5, Schaller (1912) proposes the straight line formula

$$f_r = 4{\cdot}7 - 0{\cdot}0235 \frac{L}{\kappa} \text{ t/cm.} \quad \ldots \ldots \quad (477)$$

The validity limit for Euler's formula is 86.

If a round average for $f_y$ of 55,000 lb. sq. in. be assumed, the theoretical parabolic formula becomes

$$f_r = 55{,}000 - 2{\cdot}5 \left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in.} \quad \ldots \ldots \quad (478)$$

This does not agree badly with the published experiments, giving a low average. The validity limit of Euler's formula $X_p = 105$. To represent the results of Lilly's experiments with round ends, however (Fig. 66), $\frac{L}{\kappa} > 40$, a very different formula is required :—

$$f_r = 90{,}000 - 6{\cdot}6 \left(\frac{L}{\kappa}\right)^2 \text{ lb. sq. in.} \quad \ldots \ldots \quad (479)$$

CAST IRON.—For cast-iron columns the Johnson Parabola—Euler formula is not so good, though Johnson gives the following equations :—

*Round ends:* $$f_r = 60{,}000 - \frac{25}{4}\left(\frac{L}{\kappa}\right)^2, \; X_p = 70 \quad \ldots \ldots \quad (480)$$

*Flat ends:* $$f_r = 60{,}000 - \frac{9}{4}\left(\frac{L}{\kappa}\right)^2, \; X_p = 120 \quad \ldots \ldots \quad (481)$$

The various authorities seem more or less agreed that the Rankine-Gordon formula best represents the results of experiments. The Gordon formula without doubt originated as a cast-iron column formula, and the dotted line, Fig. 72, shows to what extent the Rankine-Gordon formula,

$$f_r = \frac{80{,}000}{1 + \frac{1}{1{,}600}\left(\frac{qL}{\kappa}\right)^2} \text{ lb. sq. in.,}$$

with the usual constants, will represent Hodgkinson's experiments on specimens with round ends. Tetmajer (1896) remarks that there is no justification for the reversed curvature of the Rankine-Gordon curve (see Fig. 74), and gives the equation to a parabola about which his mean points lie grouped. Nevertheless, he says that when $\frac{L}{\kappa} > 30$ the average results are expressed with reasonable accuracy by a Rankine-Gordon curve in which $c_1 = 7{\cdot}76$ t/cm² and $c_2 = 0{\cdot}00068$. This formula applies to specimens with pointed ends. Although cast-iron

specimens are not truly elastic, in the case of position-fixed columns, when $\frac{L}{\kappa} > 80$, Euler's formula may be used. Tetmajer points out that the behaviour of the material is directly dependent on its composition. The richer in carbon,

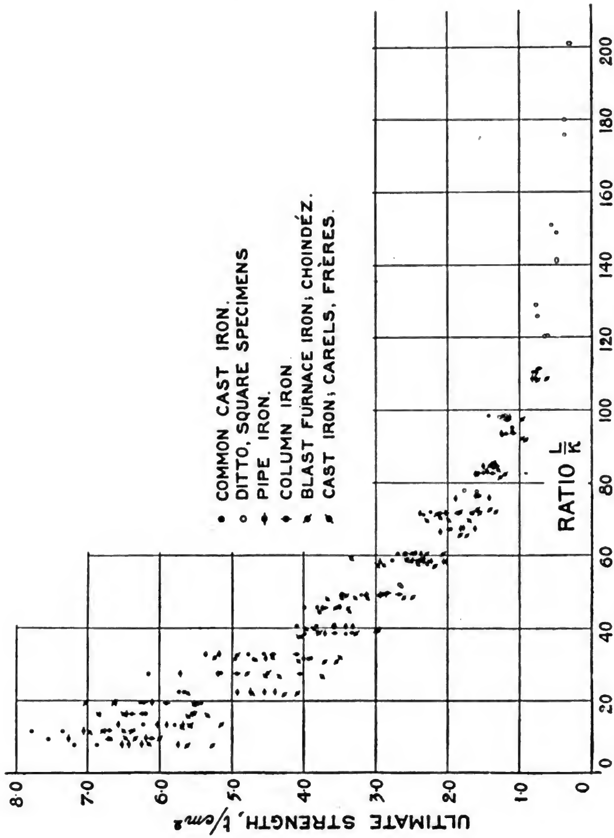



FIG. 74.—Cast-iron Specimens, Pointed Ends (Tetmajer, 1896).

the coarser and darker the grain, the greater under given conditions the permanent deformation.

Emperger (1898) also proposes to use the Rankine-Gordon formula to express the average results of Tetmajer's experiments. For the constants, he

gives: $c_1 = 7{\cdot}5$ t/cm², $c_2 = 0{\cdot}0006$. Ostenfeld (Nielson, 1911) likewise would use the same formula for cast-iron columns. He determines the constants from Tetmajer's results by the method of least squares and obtains $c_1 = 7{\cdot}76$ t/cm² and $c_2 = 0{\cdot}0007$. In British units

$$f_r = \frac{110{,}370}{1 + 0{\cdot}0007\left(\frac{L}{\kappa}\right)^2} \text{ lb. sq. in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (482)$$

This formula applies, of course, to specimens with position-fixed (pointed) ends.

All the above formulæ represent the results of experiments on carefully prepared specimens. In both Hodgkinson's and Tetmajer's tests the experimental conditions were much superior to that common in practice.

In the 1887–89 Report of the tests at Watertown Arsenal, the results of a number of tests on old hollow cast-iron mill columns are given. These had flat ends and a tapering cross section; their ultimate strength was very low, due, it appears, to spongy or otherwise defective material. Ewing (1898) gives the results of some experiments on "fair samples of the average cast-iron column used in buildings in New York city." The ends were flat. In five specimens out of ten the column was weakened by flaws or blow-holes, and the ultimate strength of all ten was low.

These results have suggested that the commonly applied Rankine-Gordon formula gives too high an ultimate strength for ordinary practical columns. It is a question to what extent this should be allowed for by the factor of safety, but in any case an indiscriminate application of the formula might lead to danger.

Burr (1898) plots the experiments of Ewing mentioned above, and suggests the straight-line formula,

$$f_r = 30{,}500 - 160\,\frac{L}{D} \text{ lb. sq. in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (483)$$

to represent the average ultimate strength. He remarks that if cast-iron columns are to be designed with a reasonable and real margin of safety, the amount of metal required dissipates any supposed economy over mild steel.

Emperger (1898), using Ewing's experiments as a basis, suggests the following formulæ for practical cast-iron columns with flat ends:—

$$f_r = \frac{7{\cdot}0}{1 + 0{\cdot}03\left(\frac{L}{\kappa}\right)^2 + \frac{\epsilon_2}{\omega}} \quad \text{t/cm}^2 \quad . \quad . \quad . \quad . \quad . \quad (484)$$

$$f_r = \frac{-3{\cdot}5}{1 - 0{\cdot}03\left(\frac{L}{\kappa}\right)^2 - \frac{\epsilon_2}{\omega}} \quad \text{t/cm}^2 \quad . \quad . \quad . \quad . \quad . \quad (485)$$

Whichever of the two formulæ gives the smaller value for $f_r$ is to be used and a factor of safety of 14 applied.

The question of a suitable factor of safety evidently depends on the formula adopted. In the light of the results of the above experiments on full-sized columns, it would appear prudent to use a factor of safety of 10 for dead loads

and 20 for live loads when applying formulæ such as (480) to (482), the constants for which were determined from carefully prepared laboratory specimens.

TIMBER.—It appears to be very doubtful whether any formula can be given which will represent the strength of timber columns with the least pretensions to accuracy. The strength of timber depends on so many factors that it would seem impossible to express them in a formula. The idea of a rational formula based on the stresses in the material may be dismissed at once as impracticable. Even an empirical formula can only be a rough mean or a lower limit to an immensely wide area (see Figs. 75 and 76), and reliance must be placed on a high factor of safety to allow for some of the most important factors on which the strength of timber depends, which factors in many experiments have been left unobserved.

According to Tetmajer (1896) the compressive strength of timber depends chiefly on its dryness, then on its nature, the portion of the tree from which it is taken, and the number and arrangement of the knots. The influence of the knots diminishes with increasing length. When $\frac{L}{\kappa} > 150$ it is negligible, provided the number is not extraordinarily great and that they are well distributed. Short specimens fail by tissue destruction, the pressing one into the other of the fibres, which always begins at a knot. Splitting or cross breaking is rare. The magnitude and direction of the deformation depend chiefly on the nature of the specimen and the number and arrangement of the knots. Even in woods free from knots the magnitude of the deflection is not proportional to the load. The behaviour of these specimens is extraordinarily varied. Long specimens $\left(\frac{L}{\kappa} > \text{about } 100\right)$, on the other hand, behave as elastic specimens; the deformation is elastic and disappears when the load is removed, tissue destruction is exceptional. To these specimens Euler's formula may be applied.

These observations have been confirmed by other experimenters. Lanza (1885) tested a number of timber specimens about 12 ft. long, from 6 to 10 in. in diameter, and with flat ends. All gave way by crushing, the ultimate strength being unaffected by the length ratio. The crushing strength per square inch varied considerably in specimens of different degrees of seasoning, and also in large and small specimens. Yellow pine posts not thoroughly seasoned nor very green failed at 4,400 lb. sq. in., whereas those of oak, which was green and knotty, but not unusually so, failed at 3,200 lb. sq. in. Nevertheless, Lanza considers that it is not safe to calculate on a higher ultimate strength in very dry specimens than in green ones.

The influence of knots is evident from the "Remarks" column in the Watertown Arsenal Tables, 1882–3, and that of imperfect seasoning in Shaler Smith's formulæ (Burr, 1883).

The 1897 Report of tests at Watertown Arsenal (1898) contains an account of some experiments on pine, spruce, and old yellow-pine specimens. It was found that the older and drier specimens gave a higher average strength than the less thoroughly seasoned ones. Their manner of failure was different. The new posts showed greater toughness of fibres, and failure was of a more or less distinctly local character. The old posts, soon after the first visible evidences of failure, fractured both by local crushing of the fibres and more or less general splitting along the grain. Knots cause loss in strength and locate the initial places of fracture.

The strongest specimens and the largest values for E were obtained from

the butt of the tree, the strength diminishing towards the top. This conclusion is confirmed in the 1904–5 Report.

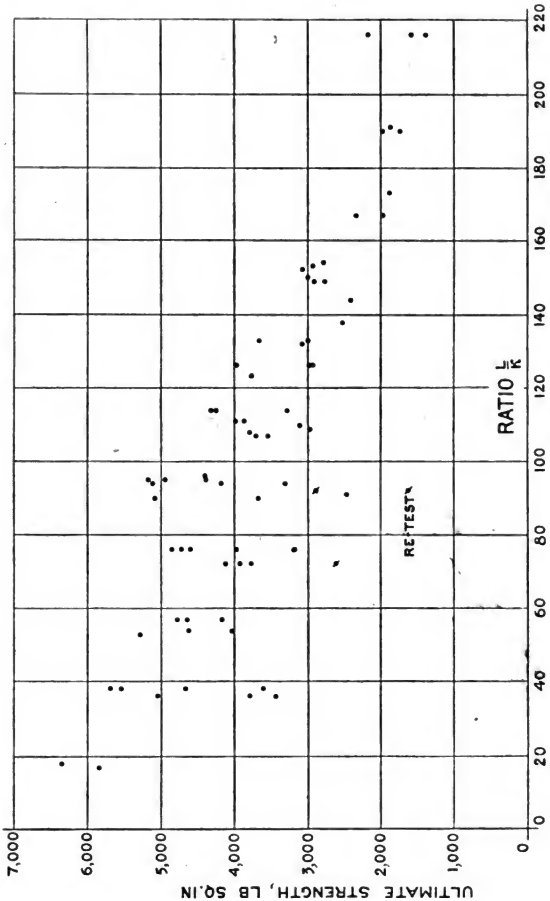



FIG. 75.—Yellow Pine Specimens, Flat Ends (Watertown Arsenal, 1882).

In the Watertown Arsenal Reports the rate of growth of the timber (rings per inch) is observed. It would appear that, other things being equal, the slower the growth the stronger the timber.

To allow for all these factors in a formula is, as has been observed, almost impossible. Probably the best that can be done is to adopt a Johnson parabola—Euler formula representing a low average of the experimental ultimate strengths; and attempt to take all other conditions into account by the use of a high factor of safety, which might be decreased if the character of the timber be known definitely and warrant such reduction.

For specimens with flat ends Johnson (1893) gives

*White pine:*

$$f_r = 2{,}500 - 0{\cdot}6\left(\frac{L}{D}\right)^2 \text{ lb. sq. in.} \qquad (486)$$

*Short-leaf yellow pine:*

$$f_r = 3{,}300 - 0{\cdot}7\left(\frac{L}{D}\right)^2 \text{ lb. sq. in.} \qquad (487)$$

*Long-leaf yellow pine:*

$$f_r = 4{,}000 - 0{\cdot}8\left(\frac{L}{D}\right)^2 \text{ lb. sq. in.} \qquad (488)$$

*White oak:*

$$f_r = 3{,}500 - 0{\cdot}8\left(\frac{L}{D}\right)^2 \text{ lb. sq. in.} \qquad (489)$$

For these formulæ the validity limit is $\frac{L}{D} = 60$. The first and third represent a low average of the Watertown Arsenal experiments (1882–3). Assuming a value $q = 0{\cdot}7$ for flat ends, the following formulæ also give a low average of the same experiments:—

*White pine:*

$$f_r = 2{,}500 - \frac{1}{10}\left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in., } X_p = 140 \qquad (490)$$

*Yellow pine:*

$$f_r = 4{,}000 - \frac{1}{8}\left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in., } X_p = 140 \qquad (491)$$

The following Johnson parabola will represent a low average of Tetmajer's experiments (1896) on timber specimens of various kinds with pointed ends:

$$f_r = 225 - \frac{1}{79}\left(\frac{qL}{\kappa}\right)^2 \text{ kg/cm}^2, X_p = 94 \qquad (492)$$

or, in British units,

$$f_r = 3{,}200 - 0{\cdot}18\left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in., } X_p = 94 \qquad (493)$$

This formula will also represent the experiments on specimens with flat ends if $q = \frac{1}{2}$.

Tetmajer himself gives the Straight line—Euler formulæ (pointed ends):

$$\frac{L}{\kappa} > 100, \quad f_p = 987\left(\frac{\kappa}{L}\right)^2 \text{ t/cm}^2, \text{ E} = 100 \text{ t/cm}^2 \qquad (494)$$

$$\frac{L}{\kappa} < 100, \quad f_r = 0{\cdot}293 - 0{\cdot}00194\,\frac{L}{\kappa} \text{ t/cm}^2 \qquad (495)$$

(see Fig. 76). These formulæ represent the average values equally as well as those proposed above. For the flat-ended specimens $q = \frac{1}{2}$.

Much experimental information regarding the ultimate strength of different

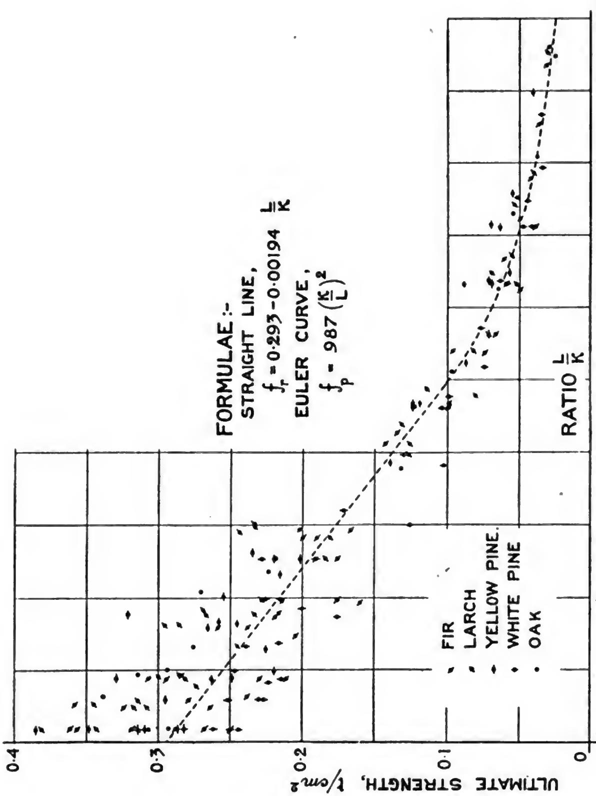



Fig. 76.—Timber Specimens, Pointed Ends (Tetmajer, 1896).

varieties of timber specimens will be found in the various Watertown Arsenal Reports, although the experiments may not be sufficiently numerous to warrant a formula being based on them.

A large factor of safety should be used with the above formulæ owing to the variable nature of the material and its tendency to rapid decay. For dead loads a value $\eta = 10$ would appear suitable.

**Ultimate Strength v. Permissible Stress Formulæ.**—This is another debatable section of the subject on which much difference of opinion has been expressed. On the one hand, it is argued that the ultimate strength is the proper criterion on which to base the safe load in a column; on the other, that a maximum permissible stress should be the determining factor. As in so many other controversial points, there is something to be said on both sides.

The advocates of the ultimate strength formulæ point out that it is possible for a long column to fail without the stress ever passing the elastic limit. Zimmermann (1886) argues that the stresses in a column are indeterminate, and therefore not a criterion on which to base strength considerations. Not only so, but the column may fail by bending to unserviceability, although the stress may be well within allowable limits. Fidler (1886) remarks that the method of taking the working flange stress as a certain fraction of the ultimate stress $f_C$ would reduce, in very long columns, the ratio of the breaking to the working load to something dangerously near to unity. The danger has also been pointed out by Bredt (1886) and many others. Bauschinger (1887) calculates the maximum stress in his specimens at the point of failure by the formula

$$f_c = \frac{W}{a}\left\{1 + \frac{y_0}{\omega}\right\},$$

and obtains such discordant results that he concludes that the maximum stress calculated in such a manner is quite inappropriate to determine the factor of safety, and that such formulæ are not suitable for determining the dimensions of columns. This he considers is an experimental verification of Zimmermann's position. Bauschinger's method of applying the formula is open to such grave objection, however, that his discordant results prove nothing more than that his method of calculation is unsuitable.

Nevertheless, Tetmajer (1896) also expresses the opinion that an experimental determination of the ultimate strength is the only safe criterion on which to base the allowable load for a column. He objects to all proposals to determine the requisite dimensions of a column on the basis of an allowable stress, or an allowable deflection, on the ground that the relation between the load and the stresses and strains produced is not definitely known. Not only so, but much valuable material does not obey the laws of elasticity.

A table given by Moncrieff (1901) is instructive. For a column in which $\frac{L}{\kappa} = 314$ an addition of 6 lb. to the applied load increased the stress from 23,200 to 53,600 lb. sq. in., and the deflection from 4 to $9\frac{1}{2}$ in. Such an example is not, however, likely in practice.

On the other hand, Findlay (1891) regards it as a false principle to base the working load on the column solely on its ultimate strength (see his second conclusion). Barth (1898), as the result of his analysis, forms the opinion that nothing is gained by determining the ultimate strength of a column, as this is of no use in determining a proper working load. He points out how different the conditions are under the working and failure loads, owing to the shape of the secant curve. Kayser's remarks (1912) may also be consulted, and Basquin's (1913).

On behalf of the ultimate strength formulæ it may be urged that the ultimate strength is easy to obtain experimentally. That the empirical formulæ are on the whole simple, and not difficult to apply. No questions appear to

arise regarding the magnitude of the eccentricity or the original curvature. The ultimate strengths of similar specimens under similar conditions will be found to agree fairly closely, while the observed deflections differ widely. Bauschinger's experiments 3028*a*–*e* (1887) are examples of this.

It is an undoubted fact that in long columns the deflection may increase to such an extent that the column would become unserviceable, and may be said to have failed, although the elastic limit is not overstepped.

On the other side, it can be objected that in no case do the experimental conditions hitherto employed remotely resemble those which exist in practice. It is well known that large initial curvatures and other imperfections are inevitable in practice. Do the empirical formulæ allow for these? Are not practical conditions inevitably inferior to laboratory conditions? Further, the conditions under the working load are quite different from those at the point of failure. It has been seen in Part II, Case II, Variation 2, that under working conditions the stress may be greater at the ends, and at the failure point greater at the centre.

It is true that the ultimate strengths of similar specimens agree fairly closely if the conditions be similar, whilst the deflections may differ considerably; but does not this point to the necessity of examining what may be the consequences of these differing deflections?

Empirical formulæ are simple and their application easy, but the application of simple formulæ to complicated cases has resulted more than once in calamity. It might be urged, further, that not a few of these very formulæ are stress formulæ used beyond the elastic limit by the artifice of substituting constants for stresses.

The question, in short, may be put: Is the safety of a practical column guaranteed by limiting the working load to a fraction of the load under which a differently situated experimental specimen failed?

Is it assured that the stress will not exceed the permissible limit provided that the load does not exceed a given fraction of the experimental ultimate strength? Jasinski (1894) answers this question by assuming that $\epsilon_1 = 0{\cdot}001\,L$, $\epsilon_2 = 0{\cdot}05\,\kappa$ to $0{\cdot}1\,\kappa$ depending on the value of $\frac{L}{\kappa}$, and that $\eta = 3{\cdot}4$; and showing that in long columns if $\frac{L}{\kappa} > 110$ to $115$ and $W = \frac{P}{\eta}$, the value of $f_c$ will not exceed $\frac{f_T}{\eta}$. Hence he concludes that Euler's formula may be used for long practical columns. Applying similar reasoning to columns in which $\frac{L}{\kappa} < 110$, but using his straight-line formula instead of Euler's curve, he concludes that $f_c$ will not exceed $\frac{f_T}{\eta}$ if

$$110 > \frac{L}{\kappa} > 58 \text{ for mild steel,}$$

$$115 > \frac{L}{\kappa} > 73 \text{ for wrought iron,}$$

and even if $\frac{L}{\kappa}$ be less than these two values, the increase in $f_c$ is not great. The

straight line may therefore be safely applied to ordinary columns although imperfections exist in them.

The values of $\epsilon_1$ and $\epsilon_2$ assumed by Jasinski may be sufficient if the practical conditions be good, but his calculation clearly points to the necessity of finding the probable values of the stress in practice, and not blindly applying empirical formulæ.

It is evident, as has, in fact, been repeatedly pointed out, that two conditions must be satisfied :—

1. The column must be stable.
2. The stress must not exceed allowable limits.

It is possible to combine the two in one formula, but it does not follow that both are satisfied either by an ultimate strength or by a permissible stress formula.

**Factors of Safety.**—The whole question, however, is bound up with that of the factor of safety. Barth (1898) pointed out that owing to the shape of the secant curve (Fig. 5, for example) a very small factor of safety would make the column abundantly secure against undue deflection. If $\eta = \frac{9}{4}$, "the deflection of an otherwise perfect column of finite length can never quite reach an amount equal to any possible actual eccentricity of the load." Even if $\eta$ be as small as 1·2, the value of $y_0$ is still restricted to $4\epsilon_2$. Kármán (1910) remarks that when $f_a$ is about equal to $\frac{4}{10} f_r$, $\Delta = \epsilon_2$.

Tetmajer (1896) suggested a factor of safety which would keep a concentrically loaded specimen in an equal stress condition, that is to say, in such a condition that the stress would be equal all over the cross section. From his experiments he concludes that in the case of mild steel no very great departure from the equal stress condition will occur if $\eta = 4$.

That the ordinary column is stable almost up to the failure point is well established by experiment. Baker (1870) found that his specimens were stable when carrying 95 per cent. of the ultimate load. Bauschinger (1887) applied a central transverse load in both directions to three of his specimens (Nos. 3028*a*, *c*, and *d*) at the time when the longitudinal load had reached one-quarter, one-half, and three-quarters respectively of its maximum value. In each case the specimen returned to its position of equilibrium on removal of the transverse load. Lilly (1908) says that with loads less than the critical load the column may be pushed in the middle, and on releasing the push it will recover itself, but will not become straight.

It follows, in fact, that provided the working load does not exceed a certain fraction of Euler's crippling load, no question of failure due to lack of stability, even in the longest columns, need be anticipated. The curves in Fig. 69 are so arranged that they apply to any value of $\frac{L}{\kappa}$. It is evident from them that provided $\frac{W}{P} < \frac{1}{2}$ no failure due to instability is possible, and in ordinary practice the factor of safety for Euler's formula is never likely to be less than, say, 4 or 5. The same thing is evident from an inspection of Figs. 5 and 15.

From this it follows that there is a minimum value of I in any column to ensure stability. Since

$$W = \frac{P}{\eta} = \frac{r^2\pi^2EI}{\eta L^2} = \frac{\pi^2EI}{\eta\,(qL)^2},$$

$$I = \frac{\eta WL^2}{r^2\pi^2E} = \frac{\eta W\,(qL)^2}{\pi^2E} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (496)$$

Provided the actual I of the cross section exceeds this value, no question of stability will arise.

The second condition for safety is complied with provided that the stress in the material does not exceed a permissible value. Here it is necessary to estimate the magnitude of the imperfections and to calculate the probable stress from a stress formula.

The column differs from all other structures, in fact, precisely on this question of the factor of safety. In a tie or beam one factor of safety only is sufficient. It will cover both accidental increases in the load and possible defects in the material. In a column the factor or factors of safety must cover not only accidental increases in the load and imperfections in the material, but also possible imperfections in the conditions or errors in the estimate of such imperfections, and also prevent undue deflection. In a beam or tie a 10 per cent. increase in the load means a 10 per cent. increase in the stress. In a column it may mean a 20 or even 50 per cent. increase in the stress. There are, in fact, in a column the three contingencies to be covered by the factors of safety. First, accidental increases in the load involving undue deflection and stress; in short, instability. This is covered, as has been seen, by limiting the load to $\frac{P}{4}$, for should the load reach double this value, the column will still be stable. Second, accidental increases in the imperfections, or underestimates of their magnitude. This may be met by making as accurate as possible an estimate of the imperfections and multiplying this estimate by a second factor of safety (see Jensen, 1908), or a sufficiently liberal estimate may be made to cover all likely increases. Fortunately a small error in the estimated value of the initial deflection or eccentricity does not much affect the result. Third, imperfections in the material. This, as has been seen, may be the most potent source of weakness. It may be met by taking low values for the material properties (Smith, 1887), or a definite estimate can be attempted of the effect of cold straightening, initial stresses, and other like imperfections (Basquin, 1913). In short, a third factor of safety is necessary.

The first factor of safety insures stability, the second and third that the stress remains within permissible limits.

Such a method of treating the problem necessitates a separate calculation for stability and strength. To this there appears to be no objection. Nevertheless, several authors have combined the two conditions in one equation. Thus Bredt (1886, 1894) multiplies both the allowable stress and the applied load by the factor of safety $\eta$ before inserting them in the formula. Under these circumstances he considers that the value $\eta = 2$ is sufficient for ordinary cases. Barth's application of the double factor of safety (1898) may also be consulted. Moncrieff (1901) proposes to allow for instability by applying the

factor of safety ($\eta = 3$) to the modulus of elasticity. He assumes for practical purposes an equivalent eccentricity, or rather a value of $\beta$,

$$\beta = \frac{e_2 v_2}{\kappa^2} = 0{\cdot}6,$$

about ten times that estimated by Marston and others as the mean value of $\beta$ in Tetmajer's experiments. Jensen (1908), it may be remarked, proposed a value $\eta = 5$ in this connexion. Further, Moncrieff limits the maximum fibre stress to about one-half the yield-point stress, or rather more than one-third of the ultimate tensile strength. In this way all three contingencies are covered in the same formula.

If an empirical formula be used, one factor of safety must, of course, cover all contingencies. Here it is not possible to take into account directly the greater imperfections probable in practice. Instability, and imperfections in the properties of material, are, to some extent at least, automatically taken into account.

Nevertheless, the fear of instability in long columns has led to the suggestion of a sliding factor of safety varying with $\frac{L}{\kappa}$. Thus Shaler Smith (Gates, 1880) proposes a factor of safety $4 + 0{\cdot}05\,\frac{L}{D}$, on the ground that the liability to imperfections increases with the length of the column. The following remark appears in the Watertown Arsenal Report (1883–4) on the experiments made on square wrought-iron specimens with pin ends: "Owing to the low transverse strength of long bars, and the serious injury which results from their deflection, it appears that bars of such a length that they fail suddenly are not so safe as their maximum resistance alone would indicate, and it would appear prudent to use a larger factor of safety as the length of the column increases."

Emperger (1897) treats the matter in a slightly different way. He plots the results of experiments on flat-ended columns and determines a formula of the Rankine-Gordon type to represent the average or mean values. In this formula the constant $c_2 = 0{\cdot}000032$. For practical use, however, he proposes to use a lower limit formula in which $c_2 = 0{\cdot}00005$, thus allowing for practical inaccuracies. These inaccuracies, he argues, will increase with the value of $\frac{L}{\kappa}$, and therefore the factor of safety should increase also. Actually, the second equation will give a factor of safety of $4\left\{1 + 0{\cdot}56\,c_2\left(\frac{L}{\kappa}\right)^2\right\}$ under conditions in which the first equation would give a factor of safety of 4.

On the other hand, the shape of the $\left(f_r - \frac{L}{\kappa}\right)$ diagram, with its great increase in width as the value of $\frac{L}{\kappa}$ diminishes, suggests that if an average curve be adopted, the factor of safety should allow for the possible variation in the ultimate strength. It has been suggested previously that the value of $\eta$ should be made a function of the ratio

$$\frac{\text{upper limit load} - \text{lower limit load}}{\text{mean load}}.$$

Even if the lower limit line be adopted, it does not follow that the experimental conditions will be the same as those in practice. Nevertheless, a factor of safety of 4 is usually considered sufficient for columns of ordinary length of ductile material such as mild steel, if a lower limit line be adopted; which should be increased to 5 if a high average curve be used. For columns coming within the range of Euler's formula $\left(\frac{q\mathrm{L}}{\kappa} > 100\right)$, the factor of safety should never be less than 5 whatever be the formula used.

In the case of cast-iron and timber columns, a factor of safety of 5 is undoubtedly insufficient. These materials exhibit such large variations in quality, and in cast-iron columns the probability of imperfect castings, blow-holes, etc., is so great, that the factor of safety for dead loads to be used with formulæ derived from experiments on carefully prepared laboratory specimens should be double that used for ductile materials under similar circumstances.

It is the custom in many quarters to limit the value of $\frac{\mathrm{L}}{\kappa}$ used in practice, so that no question of stability, and therefore of the employment of Euler's formula, may arise. It is probably wise, in the case of important columns, to limit the value of $\frac{q\mathrm{L}}{\kappa}$ to 120, but there appears to be no reason why a column of any reasonable proportions could not be successfully designed.

To allow for the effect of varying and alternating loads on a column, the maximum stress permissible under a dead load should be reduced in the same ratio as in the case of a tie. It does not appear necessary to increase the factor of safety adopted to ensure stability, for, as has been seen, the load may increase from $\frac{\mathrm{P}}{5}$ to $\frac{\mathrm{P}}{2}$, or even more, without impairing the stability of the column.

Regarding this latter point, Moncrieff (1901) says that a most interesting feature of the table quoted, p. 241, is the theoretic assurance which it gives as to the capacity of long columns to resist fatigue, even when loaded nearly up to the crippling point. He remarks that the specimen in the example chosen would be quite uninjured by an infinite number of loadings within 10 lb. of its ultimate strength, if the load were applied without impact.

Claxton Fidler, on the other hand, would determine the necessary area by a column formula (Rankine's), assuming the load to be a static one; and alternatively by an alternating stress formula, neglecting any tendency to buckle. The greater of the two areas found in this way is to be used. He does not consider it necessary to reduce the coefficients in the column formula to allow for variations in the load.

Several writers have, however, combined column and alternating stress formulæ. The work of Hanna (1910) and Orrell (1910) may be consulted in this connexion.

**Methods of Design.**—Given that a solid column is to be of a certain length, of a given material, and to carry a certain load under certain definite conditions of loading, to find the requisite dimensions:—

If the column be position-fixed, the load concentric, and it is desired to

adopt a solid cross section, an empirical formula may be used. Suppose, for example, the Johnson parabola be chosen [equation (469)]

$$f_r = c_1 - c_2 \left(\frac{L}{\kappa}\right)^2.$$

If the given load be W, the ultimate strength of the column must be $\eta$W, where $\eta$ is the required factor of safety. Knowing the ultimate strength, it is necessary to find $f_r$ before the area can be determined. But $f_r$ is a function of $\kappa$, and is unknown until the shape and size of the cross section is known. To solve the problem, therefore, it appears necessary to proceed by a trial and error method. Assume a cross section, find $\kappa$ and $f_r$, and hence a value for $a$. This will probably differ from that assumed. A new trial must then be made, and the process repeated until the area found agrees with the area assumed.

This trial and error method is awkward, but would have to be gone through whatever the empirical formula used, were it not for a device due to Asimont (1876–7). Its application to the Johnson parabola was made by Ostenfeld (1898). Let $a = g\kappa^2$. If the column were very short, the area required would be

$$a_0 = \frac{W\eta}{c_1} \qquad \text{(497)}$$

But the formula may be written

$$\frac{W\eta}{a} = c_1 - c_2 \cdot \frac{L^2}{a} g.$$

Hence

$$a = a_0 + \frac{c_2}{c_1} g \cdot L^2 \qquad \text{(498)}$$

where

$$\frac{c_2}{c_1} = \frac{f_y}{4\pi^2 E} = \frac{1}{30{,}000},$$

approximately, for mild steel. All the quantities in this equation are known except $a$, which may, therefore, be found directly. The value of $g$ may easily be determined from a cross section similar to the one which it is proposed to use. Tables of values for $g$ have been given by Asimont, Fidler, Ostenfeld, and others.

Asimont actually applied his method to the Rankine-Gordon formula. Fidler also (1887) has used the same device with the Rankine-Gordon formula, and has given tables of multipliers whereby the area may quickly be found. The straight-line formula does not lend itself so readily to this method, as the equation for $a$ becomes a quadratic. The apparent simplicity of this formula is, therefore, a little delusive. A device of Brik's (1911) to simplify the use of the Tetmajer—Euler equations may, perhaps, be mentioned.

It is possible, of course, by the use of any empirical formula, to plot curves for particular cross sections showing the variation in the ultimate strength with increasing length, and from such curves to pick out a suitable cross section. The labour involved in making the curves is, however, considerable.

Instead of a purely empirical formula, semi-rational formulæ like Fidler's (1887) or Moncrieff's (1901) may be adopted. These expressions are somewhat complicated, and tables and curves are necessary in practice.

A third alternative is to base the design solely on a stress formula. The

work of Alexander (1912) and Kayser (1912) may be quoted as typical examples of this method. The former proposes to find the worst direction for the eccentricity, which introduces a complication into the work and appears hardly necessary, however well justified. The vast majority of specimens in a testing machine deflect in the direction of the least radius of gyration or thereabouts, unless the end conditions be very different in the two directions.

If the load have an intentional eccentricity, a simple empirical formula like the Johnson parabola can no longer be used. It is, nevertheless, possible to use the eccentricity form of the Rankine-Gordon formula, and even to apply Asimont's device thereto (Ostenfeld, 1898).

A method due to Tetmajer (1896) should be mentioned. From his experiments on eccentrically loaded specimens, Tetmajer came to the conclusion that such specimens failed when the stress in the extreme fibres was equal to the ultimate tensile strength of the material divided by an empirical coefficient $\mu$

$$f_c = \frac{f_T}{\mu} = \frac{\eta W}{a}\left\{1 + \frac{v_2 y_0}{\kappa^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (499)$$

where W is the working load.

Having given the value of $\epsilon_2$, the value of $y_0$ for the load $\eta$W can be found from the eccentricity formula. The method is a trial and error one, an approximate cross section is assumed and the value of $f_c$ found. If this exceed $\frac{f_T}{\mu}$, a modification must be made to the area and a fresh value for $f_c$ calculated. As an average value for $\mu$, applying to mild-steel specimens, Tetmajer gives $\mu = 1{\cdot}37$. An application of the same proposal to the eccentricity form of the Rankine-Gordon formula was made by Ostenfeld (1902), who determined values of $\mu$ to suit.

If it be preferred to work on a stress basis solely, the eccentricity formula may be used. This, again, has to be solved by trial and error, unless use be made of curves such as those proposed by Smith (1887).

It is needless to add that the eccentricity formula can be applied to columns intended to be concentrically loaded if a value for $\epsilon_2$ equal to the probable unintentional eccentricity be adopted. It is equally possible to apply formulæ such as Moncrieff's (1901) and Alexander's (1912) to eccentrically loaded columns if, for the unintentional eccentricity used in these formulæ, be substituted the sum of the intentional and unintentional eccentricities.

For columns with an initial curvature, Ayrton and Perry's formulæ (1886), or those of Part II, Case I, Variation 2, may be used.

As an alternative to the above solutions to the problem of a position-fixed column, whether concentrically or eccentrically loaded, initially straight or initially curved, the author would propose the following method, which he considers to be as simple as any, and in many ways preferable. In the first place, it is necessary to guard against failure by instability. This, as has been seen (p. 244), implies a minimum value for I, equation (496):

$$I = \frac{\eta\, WL^2}{\pi^2 E}.$$

This minimum value is easily determined from the given conditions. The value of $\eta$ should not be less than 5. In the second place, the stress should

not exceed an allowed limit $f_c$. This limit, in ductile materials, should be 10 to 20 per cent. less than the corresponding stress allowed in ties under similar circumstances, in order to allow for reductions in the strength of the material due to shop treatment; allowance should also be made in the usual way for any fluctuations in the load. Having fixed $f_c$, the necessary area can be found from equation (94):

$$a = \frac{W}{f_c}\left\{1 + 1{\cdot}3\,\frac{v_2}{\kappa^2}\,(\epsilon_1 + \epsilon_2)\right\}.$$

To solve this equation it is necessary to estimate $\epsilon_1$ and $\epsilon_2$ [see pp. 143 and 154 and equations (420) and (421)], the value of $\epsilon_2$ will include, of course, any intentional eccentricity. The maximum value of $\frac{v_2}{\kappa^2}$ must be determined from a section of a type and size *similar* to the one it is desired to adopt. No great error will result if the first estimate is not very accurate. Knowing thus the minimum values of I and $a$, a suitable section of the requisite type can be at once chosen. It is now desirable to check the value assumed for $\frac{v_2}{\kappa^2}$ and to recalculate if necessary. If the ratio $\frac{W}{P}$ be considerably smaller than $\frac{1}{5}$, it is better to use equation (93A) in place of (94).

If it should happen that a section of a definite depth $2v_2$ is required, the minimum area can be at once obtained from equation (102).

Having determined the cross section, should the importance of the column be sufficient, an exact value for $f_c$ may be found from equation (91), taking into account all the possible imperfections in the column. This, as a rule, is unnecessary.

This method is applicable only to ductile materials. The author considers that nothing is gained in the case of cast-iron and timber columns bysupplanting the empirical formulæ discussed on pp. 234 and 237.

If the column be position- and direction-fixed instead of merely position-fixed, the usual procedure is to select a value for $q$ and use the empirical formulæ for position-fixed columns, substituting $q$L for L. This is not altogether satisfactory for the reasons given on p. 172. In any case the value taken for $q$ in practical work should be greater than the theoretical value 0·5, a value from 0·7 to 0·8 representing more nearly the actual conditions.

For ductile materials a method analogous to that last proposed for position-fixed columns is much to be preferred.

To guard against failure by instability, it is first necessary to determine the minimum value of I.

$$I = \frac{\eta W (qL)^2}{\pi^2 E} \qquad \qquad (496)$$

The value of $\eta$ should not be less than 5. Next, the permissible value for the maximum stress $f_c$ must be determined. For dead loads this should be from 10 to 20 per cent. less than the safe tensile stress under similar circumstances. If the load fluctuate in value, allowance should be made for this in the usual

way. Having fixed $f_c$, to find the necessary area equations (182) and (187) must be solved.

$$a = \frac{W}{f_c}\left[1 + \frac{\epsilon_1 v_2}{\kappa^2}\left\{1{\cdot}44\,k - 1\right\}\right] \quad . \quad . \quad . \quad . \quad . \quad (182)$$

$$a = \frac{W}{f_c}\left[1 + 0{\cdot}78\,\frac{\epsilon_1 v_1}{\kappa^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

The larger of the two values for $a$ must, of course, be adopted. Regarding the value of $k$, probably the best that can be done in the absence of more definite information is to adopt the values given in the table on p. 72. If, however, the degree of imperfection in the direction-fixing be of the order there assumed, it is more convenient to determine the area at the centre from equation (205A) instead of equation (182)

$$a = \frac{W}{f_c}\left[1 + 0{\cdot}76\,\frac{\epsilon_1 v_2}{\kappa^2}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205\text{A})$$

which, as a rule, will give a smaller value of $a$ than equation (187), and need not then be further considered. The value for $\epsilon_1$ can be estimated [equations (422) and (423)], no question regarding eccentricity arises, and a value for $\frac{v_2}{\kappa^2}$ may be obtained from a section of a type and size similar to the one it is desired to adopt. In determining the value of $\frac{v_2}{\kappa^2}$, it is necessary to take into consideration the worst direction in which the column may deflect as a direction-fixed member. If the member is equally free to deflect in all directions, the maximum value of $\frac{v_2}{\kappa^2}$ should be used, but it is sufficient to take the least value of $\kappa$ and the larger value of $v_1$ or $v_2$ which corresponds—there is no necessity to seek the minimum radius of the core. No great error will result if the first estimate is not very accurate. Having obtained thus the minimum values of I and $a$, a suitable section of the type required may be chosen. It is now desirable to check the value assumed for $\frac{v_2}{\kappa^2}$ and to recalculate if necessary. If the value of $\frac{W}{P_2}$ be much less than $\frac{1}{5}$, the more exact equations (181)* and (186) should be used to find the necessary area, and it is then necessary to calculate the value of $a$ both at the middle and at the ends, and to adopt the larger value.

In important columns an exact value for $f_c$ at the centre of the column may be found from equation (175), and at the ends of the column from equation (177), in which equations all possible imperfections are taken into account.

In a lattice-braced column with position-fixed ends, given the overall length of the member, the working load, and the safe stress, to design the column.

The safe stress should be 20 per cent. less than that allowable in a tension member under similar conditions. This fixes $f_c$. First assume that all the

* Or, alternatively, equation (205).

imperfections tend to produce bending in the plane $xz$ (Fig. 58). In this direction the column may be treated as a solid column. The minimum value of the moment of inertia of the column as a whole about the axis $yy$, from conditions of stability [equation (496)], is

$$I_s = \frac{\eta WL^2}{\pi^2 E}.$$

The value of $a_0 = \frac{W}{f_c}$, from which an estimate of the value of $a$ can be made; or, if preferred, a direct estimate of $a$ can be obtained from equation (498). From the known value of $I_s$ and the estimated value of $a$, a provisional section can be selected. The true value of $a$ is then found from equation (94):

$$a = \frac{W}{f_c}\left\{1 + 1{\cdot}3\,\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\};$$

or, since in lattice-braced columns the value of $\frac{L}{\kappa}$ and, therefore, of $\frac{W}{P}$ is usually small, it is preferable to use the more exact expression, equation (93A):

$$a = \frac{W}{f_c}\left\{1 + \left(1 + \frac{3W}{2P}\right)\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\}.$$

It may be necessary to revise the selected section to suit this value of $a$, which should now be done, but a small excess of area is not unadvisable.

Consider next bending in the plane $xy$. Presuming that nothing is fixed regarding the value of $h$ or D, it is necessary to consider the stress in an elementary flange column (Fig. 35). The maximum stress $f_c$ in such a column is given approximately by the equation

$$f_c = \frac{F_c}{a_2}\left\{1 + 1{\cdot}3\,\frac{v_2''}{(\kappa'')^2}(\epsilon_1'' + \epsilon_2'')\right\};$$

or, more exactly, by equation (311):

$$f_c = \frac{F_c}{a_2}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\}.$$

In this equation all the factors* are known or can be found except $F_c$ and $P''$. $P''$ is Euler's crippling load for the elementary flange column; its value is

$$P'' = \frac{\pi^2 EI_2}{j^2}$$

It is necessary, therefore, to assume values for $j$ and $F_c$. The value of $j$ can be estimated roughly from the general dimensions of the column, its exact

* It is preferable to take $\epsilon_1''$ as $\frac{j}{375}$ instead of $\frac{j}{750}$, for the elementary flange column cannot be looked upon as a properly straightened column.

value from the present point of view being as a rule unimportant. $F_c$ may be given a value slightly greater than $\frac{W}{2}$.

Substituting these values in equation (311), a value for $f_c$ will be obtained. This may be greater or less than the allowed stress; however, by direct proportion a new value for $F_c$ can be found which will produce a stress $f_c$ in the material equal to the allowed limit. It remains to find a value for $h$ such that the force on the elementary flange column will not exceed the new value for $F_c$. From equation (309)

$$F_c = W\left[\frac{1}{2} + \left(1 + \frac{3W}{2P}\right)\frac{\epsilon_1 + \epsilon_2}{h}\right].$$

In this equation the values of $\epsilon_2$ and P are directly dependent on $h$, and it will be found that a considerable variation in $h$ produces a comparatively small variation in $F_c$. It is necessary, therefore, to give an exact solution to this equation. Now $P = \frac{\pi^2 EI_y}{L^2} = \frac{\pi^2 Eah^2}{4L^2}$, hence the factor

$$\left(1 + \frac{3W}{2P}\right) = \left(1 + \frac{6W}{\pi^2 Ea}\cdot\frac{L^2}{h^2}\right).$$

Further, $\epsilon_1 = \frac{L}{750}$, and if $\epsilon_2$ from equation (421) is equal to $\frac{L}{1000} + \frac{h}{20} + \frac{h}{160}$, the expression for $F_c$ becomes

$$\left(\frac{F_c}{W} - \frac{1}{2}\right) = \left\{1 + \frac{6W}{\pi^2 Ea}\cdot\frac{L^2}{h^2}\right\}\left\{\frac{L}{h}\left(\frac{1}{1000} + \frac{1}{750}\right) + \left(\frac{1}{20} + \frac{1}{160}\right)\right\}.$$

Let $\frac{L}{h} = X$, then

$$\frac{40\left(\frac{F_c}{W} - \frac{1}{2}\right)}{1 + \frac{6W}{\pi^2 Ea}\cdot X^2} = 2{\cdot}25 + \frac{7}{75}X \quad . \quad . \quad . \quad . \quad . \quad (500)$$

This is a cubic equation, and is most easily solved graphically. In ordinary calculations the factor $\left\{1 + \frac{6W}{\pi^2 Ea}X^2\right\}$ will not differ much from unity, and, therefore, as a first approximation

$$\frac{L}{h} = X = \frac{75}{7}\left\{\frac{40F_c}{W} - 22{\cdot}25\right\} \quad . \quad . \quad . \quad . \quad (501)$$

This gives a value for $\frac{L}{h}$ somewhat too large, and, therefore, a value for $h$ somewhat too small. An exact solution to equation (500) is, however, not difficult. The value of $h$ found by this method may or may not be suitable.

If it be not suitable, a new value for $a$ may be chosen, and a fresh attempt made. In certain cases the value of $h$ may be fixed by the conditions of the problem, or it may be preferable to choose a value of $h$ and find the necessary area of cross section. In this case, having fixed a suitable cross section to resist flexure in the plane $xz$, the value of $F_c$ due to bending in the plane $xy$ is determined from equation (309):

$$F_c = W \left\{ \frac{1}{2} + \left(1 + \frac{3W}{2P}\right) \frac{(\epsilon_1 + \epsilon_2)}{h} \right\}$$

Knowing $F_c$, the requisite value for $a_2$ can be found from equation (311):

$$a_2 = \frac{F_c}{f_c} \left\{ 1 + \frac{v_2''}{(\kappa'')^2} \left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'') \right\}$$

and a suitable section chosen (see footnote, p. 251).

If the flange itself be built up, and composed partly of flange plates, it is necessary to modify the above equation. This matter is discussed at the end of the section on "Bridge Compression Members" following.

The maximum shearing force on the column is given by equation (445):

$$Q_{max} = \frac{\pi W}{2L}\left(\epsilon_2 + \frac{32\epsilon_1}{\pi^2}\right).$$

This equation is strictly true only at the limit $\frac{W}{P} = \frac{1}{4}$, but it is quite accurate enough for all practical purposes. The lattice bracing everywhere should be designed to carry this load; and when determining the size of the lattice bars and the rivets in them, unless the load on the column be a dead load, suitable reductions in the working stresses should be made to allow for its varying nature.

BRIDGE COMPRESSION MEMBERS.—In a bridge the maximum and minimum loads will be known and the length of the column. Unless pin joints are used, the ends of the member will be firmly riveted to the flanges, and may be treated as imperfectly direction-fixed. Usually the dimensions of the flange will fix the overall depth D of the column.

Knowing the maximum and minimum loads, the safe working stress $f_c$ should be found by the Launhardt-Weyrauch or Claxton Fidler formulæ. The safe working stress in compression should be 20 per cent. less than that proper in tension, to allow for reductions in the quality of the material. Since the column is imperfectly direction-fixed, the value of $q$ may be taken as 0·78.

First assume that all the imperfections tend to produce flexure in the plane $xz$ (Fig. 58). In this direction the column may be treated as a solid column. From equation 496 the minimum value of the moment of inertia of the column as a whole about the axis $yy$, if $\eta = 5$, is

$$I_z = \frac{5W\,(qL)^2}{\pi^2 E}.$$

The value of $a_0 = \frac{W_{max}}{f_c}$, from which the value of $a$ can be estimated, or a value for $a$ can be obtained from equation (498). Knowing $I_z$ and $a$, a

provisional section may be chosen. The elements of the section picked can be calculated, and a truer estimate of the value of $a$ obtained from equation (181)

$$a = \frac{W}{f_c}\left[1 + \frac{2k\epsilon_1 v_2}{\kappa^2}\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{1}{\pi^2}\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

The value of $k$ can be obtained from Fig. 15, which should be plotted to a larger scale from the table on p. 72 ; $\epsilon_1$ is given by equation (422):

$$\epsilon_1 = \frac{L}{750} + \frac{1}{10}\frac{a_1\bar{v}_1}{a}.$$

It is worth while using equation (181) rather than the upper limit equation (182), for the value of $\frac{W}{P_2}$ is usually small. Alternatively, equation (205) may be used.

Equation (181), however, gives the stress at the centre of the column, and it is possible, particularly when the ratio of $\frac{W}{P_2}$ is small, that the maximum stress may occur at the ends. In finding the area at the ends, $k$ should be taken equal to unity, for this assumption gives the largest value of the stress at the ends, and, therefore, the greatest value for $a$. If $k \doteq$ unity, from equation (186)

$$a = \frac{W}{f_c}\left[1 + \frac{\epsilon_1 v_1}{\kappa^2}\left\{0{\cdot}66 + 0{\cdot}58\frac{W}{P_2}\right\}\right]$$

As a rule $v_1 = v_2$. The larger of the two values of $a$ must be used, and if necessary the provisional section modified to suit.

Next consider bending in the plane $xy$. Since the value of D is fixed, the value of $h$ can be determined, and, therefore, $I_y$ and $P_2$ for the column as a whole. The value of $F_c$, the maximum force on an elementary column at the centre of the main column, can be found from equation (322):

$$F_c = \frac{W}{h}\left[\frac{h}{2} + 2k\epsilon_1\left\{0{\cdot}17 + 0{\cdot}26\frac{W}{P_2} + \frac{P_2}{\pi^2 W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

As before, $k$ may be obtained from Fig. 15 ; $\epsilon_1$ is given by equation (423):

$$\epsilon_1 = \frac{L}{750} + \frac{h}{40}.$$

At the ends of the main column the maximum force on an elementary flange column, if the worst assumption, $k = 1$, be made, is given by equation (331):

$$F_c = W\left[\frac{1}{2} + \frac{\epsilon_1}{h}\left\{0{\cdot}66 + 0{\cdot}58\frac{W}{P_2}\right\}\right]$$

This equation will be found to give larger values of $F_c$ than equation (322)

if the value of $\frac{W}{P_2}$ is small. The larger value of $F_c$ should be used in what follows.

Since $h$ is known, and the angle of the bracing can be fixed, usually at 45°, the value of $j$ can be found. Hence Euler's crippling load $P''$ for the elementary flange column can be found, and also the ratio $\left(1 + \frac{3F_c}{2P''}\right)$. Then the required area to resist bending in this direction is given by equation (311)

$$a_2 = \frac{F_c}{f_c}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\}$$

In this equation $\epsilon_1''$ should be taken as $\frac{j}{375}$, i.e. double the value usual in properly straightened columns, for the elementary column cannot be looked upon as a straightened specimen. The value of $\epsilon_2$ can be found from equation (420) in the usual way.

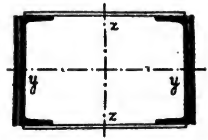


FIG. 77.

If the value of $a_2$ obtained from equation (311) differ from the previous estimate, the proposed section must be modified to suit.

In such columns the flange is sometimes constructed partly of flange plates (Fig. 77). In this case the value of $f_c$ must be modified to allow for tertiary flexure. This is most conveniently done by the application of equation (340), by which the maximum allowable stress $f_c$ is reduced to $f_c'''$ where

$$f_c''' = \frac{f_c}{1 + \frac{p}{80t}}.$$

$p$ is the longitudinal pitch of rivets and $t$ the thickness of the flange plate. Equation (311) giving the value of $a_2$ then becomes

$$a_2 = \frac{F_c}{f_c'''}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\} \quad . \quad . \quad . \quad (502)$$

In this equation it should be noted that the value of $\frac{v_2''}{(\kappa'')^2}$ should be calculated for the side of the section on which the flange plate occurs (see Fig. 36). It is obvious that tertiary flexure can only occur in the flange plate when that plate forms the concave side of the elementary flange column, and in such sections as that shown in Fig. 77 the stress will be less on the flange plate side of the section than on the other. It is necessary, in fact, to consider two cases, (i) with the flanges of the channel and (ii) with the flange plate forming the concave side of the elementary flange column respectively [equations (311) and (502)]. The larger value of $a_2$ found by these formulæ is the required area.

The maximum shearing force on the column is given by equation (447),

$$Q_{max} = W\frac{4k\epsilon_1}{L},$$

and the lattice bracing everywhere should be designed to carry this load. Normally it is difficult to make the bracing sufficiently light, but in determining the size of the lattice bars and the rivets in them the varying nature of the load, and the necessary reduction of the stress which it entails, should be allowed for.

## EXAMPLES

*An 8 in. × 6 in. × 35 lb. British Standard Beam No. 14 of mild steel, 100 in. long, acts as a position-fixed column. The load is intended to be concentric, what ought its magnitude to be?*

Apply first the Johnson parabola, equation (474):

$$f_r = 40{,}000 - \frac{4}{3}\left(\frac{qL}{\kappa}\right)^2 \text{ lb. sq. in.}$$

$\kappa$ minimum for the section $= 1{\cdot}32$ inches, $q = 1$, and hence $\frac{qL}{\kappa} = \frac{100}{1{\cdot}32} = 75{\cdot}8$.

Therefore, $f_r = 40{,}000 - \frac{4}{3}(75{\cdot}8)^2 = 32{,}350$ lb. sq. in. Allowing a factor of safety of 4, the safe load

$$W = \frac{32{,}350 \times 10{\cdot}3}{4 \times 2240} = \mathbf{37{\cdot}2} \text{ tons.}$$

where $a$, the area of the section, $= 10{\cdot}3$ sq. in.

Checking this result by the usual Rankine-Gordon formula (p. 224),

$$W = \frac{1}{\eta}\frac{48{,}000\, a}{1 + \frac{4}{30{,}000}\left(\frac{L}{\kappa}\right)^2},$$

the resulting value for W is, if $\eta = 4$,

$$W = \frac{1}{4}\cdot\frac{48{,}000 \times 10{\cdot}3}{1{\cdot}766 \times 2240} = \mathbf{31{\cdot}3} \text{ tons.}$$

As an alternative to the application of an empirical formula, the rational method proposed on p. 248 will next be used. In the first place, from considerations of stability,

$$W = \frac{P}{5} = \frac{1}{5}\frac{\pi^2 EI}{L^2}$$

Now I minimum = 17·95 in.$^4$, hence

$$W = \frac{1}{5} \times \frac{\pi^2 \times 13{,}000 \times 17{\cdot}95}{100^2} = \mathbf{46} \text{ tons.}$$

Secondly, from equation (94) for the maximum stress,

$$W = \frac{f_c a}{1 + 1{\cdot}3 \frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)}.$$

In this equation, if $\eta = 4$, $f_c = \frac{28}{4} = 7$ tons sq. in., less, say 20 per cent., to allow for reduction in the strength of the material; therefore $f_c = 5{\cdot}6$ tons sq. in. The area $a = 10{\cdot}3$ sq. in., and from the properties of the cross section $\frac{v_2}{\kappa^2} = \frac{3}{(1{\cdot}32)^2}$. The probable initial deflection $\epsilon_1 = \frac{L}{750} = \frac{100}{750} = 0{\cdot}13$ in. The probable eccentricity of loading $\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a}$ [equation (420)]. Now $a_1 = \frac{a}{2}$, and $\bar{v}_1 = 1{\cdot}1$ in. approximately. Hence, $\epsilon_2 = \frac{100}{1000} + \frac{1}{5} \cdot \frac{1{\cdot}1}{2} = 0{\cdot}10 + 0{\cdot}11$ in., and $\epsilon_1 + \epsilon_2 = 0{\cdot}13 + 0{\cdot}10 + 0{\cdot}11 = 0{\cdot}34$ in. Inserting these values in the formula,

$$W = \frac{5{\cdot}6 \times 10{\cdot}3}{1 + 1{\cdot}3 \frac{3}{(1{\cdot}32)^2} \times 0{\cdot}34} = \frac{5{\cdot}6 \times 10{\cdot}3}{1{\cdot}76} = \mathbf{32{\cdot}8} \text{ tons.}$$

Since this figure is less than $\frac{P}{5}$, it represents the safe load on the column. It is of interest to observe that the value of the denominator in the equation is almost exactly equal to that in the Rankine-Gordon formula. If now the more exact expression, equation (93A),

$$W = \frac{f_c a}{1 + \frac{v_2}{\kappa^2}\left\{(\epsilon_1 + \epsilon_2)\left(1 + \frac{3W}{2P}\right)\right\}}$$

be used instead of equation (94), $\frac{W}{P} = \frac{32{\cdot}8}{46 \times 5} = 0{\cdot}143$ and

$$W = \frac{5{\cdot}6 \times 10{\cdot}3}{1{\cdot}71} = \mathbf{33{\cdot}7} \text{ tons,}$$

or, under a load of 32·8 tons, the actual stress

$$f_c = \frac{32{\cdot}8}{10{\cdot}3} \times 1{\cdot}71 = \mathbf{5{\cdot}46} \text{ tons sq. in.}$$

It is evident, therefore, that the error introduced by using equation (94) is small.

*A column of the same section as the above (B.S.B. No. 14) is 128 in. long, and intended to be position- and direction-fixed at its ends. What is the safe load?*

Assuming a value of $q = 0{\cdot}78$ (Fig. 15) for imperfectly direction-fixed ends, the free length is

$$qL = 128 \times 0{\cdot}78 = 100 \text{ in.}$$

and hence, as in the previous example, the safe load by the parabolic formula is

$$W = \mathbf{37{\cdot}2} \text{ tons.}$$

If, again, the length $qL$ is substituted in the Rankine-Gordon formula given above,

$$W = \mathbf{31{\cdot}3} \text{ tons}$$

as before. On the other hand, if the ordinary form of that equation for direction-fixed ends be applied

$$W = \frac{1}{\eta}\,\frac{48{,}000\,a}{1 + \dfrac{1}{30{,}000}\left(\dfrac{L}{\kappa}\right)^2},$$

the resulting value of W is

$$W = \frac{1}{4}\cdot\frac{48{,}000 \times 10{\cdot}3}{1{\cdot}314 \times 2240} = \mathbf{42{\cdot}0} \text{ tons.}$$

Here the direction-fixing is assumed to be perfect.

Applying next the rational method (p. 249), from stability considerations [equation (496)]

$$W = \frac{P}{5} = \frac{1}{5}\,\frac{\pi^2 EI}{(qL)^2} = \mathbf{46} \text{ tons}$$

as before. From equation (205A) for the stress at the centre of the column,

$$W = \frac{f_c a}{1 + 0{\cdot}76\,\dfrac{\epsilon_1 v_2}{\kappa^2}}.$$

Here, as in the first example, $f_c = 5{\cdot}6$ tons sq. in., $a = 10{\cdot}3$ sq. in., and $\dfrac{v_2}{\kappa^2} = \dfrac{3}{(1{\cdot}32)^2}$. The initial deflection $\epsilon_1 = \dfrac{L}{750} + \dfrac{1}{10}\,\dfrac{a_1 \bar{v}_1}{a}$ [equation (422)] $= 0{\cdot}13 + 0{\cdot}06 = 0{\cdot}19$ in., and therefore,

$$W = \frac{5{\cdot}6 \times 10{\cdot}3}{1 + 0{\cdot}76 \times \dfrac{3}{(1{\cdot}32)^2} \times 0{\cdot}19} = \mathbf{46{\cdot}2} \text{ tons.}$$

If the direction-fixing were perfect, the maximum stress would occur at the ends of the column, and equation (187) would be applicable. This becomes

$$W = \frac{5{\cdot}6 \times 10{\cdot}3}{1 + 0{\cdot}78 \times \dfrac{3}{(1{\cdot}32)^2} \times 0{\cdot}19} = \mathbf{45{\cdot}7} \text{ tons.}$$

The smallest of the three values, W = 45·7 tons, is the safe load on the column.

Comparing the results obtained in these two examples,

*Position-fixed Column,* 100 *in. long.*

Safe load:

| | | |
|---|---|---|
| | 37·2 tons | Johnson parabola. |
| | 31·3 „ | Rankine-Gordon. |
| { | 46·0 „ | Euler. |
| { | 32·8 „ | Stress formula (94). |

*Position- and Direction-fixed Column,* 128 *in. long.*

Safe load:

| | | | |
|---|---|---|---|
| | 37·2 tons | Johnson parabola . | $q = 0{\cdot}78$ |
| | 31·3 „ | Rankine-Gordon . . | $q = 0{\cdot}78$ |
| | 42·0 „ | „ „ . . | $q = 0{\cdot}5$ |
| { | 46·0 „ | Euler . . . . . . | $q = 0{\cdot}78$ |
| { | 46·2 „ | Stress formula (205A) . | $q = 0{\cdot}78$ |
| { | 45·7 „ | „ „ (187) . | $k = 1{\cdot}0$ |

The smallest of the loads bracketed together is the safe load by the rational method.

It should be remembered, when comparing these results, that the safe stress when $\frac{L}{\kappa} = O$, by these formulæ, is

| | |
|---|---|
| Johnson parabola . | = 10,000 lb. sq. in. |
| Rankine-Gordon . | = 12,000 „ „ „ |
| Rational method . | = 12,540 „ „ „ |

*A column of angle section,* 5 *ft. long, is required to carry a load of* 5 *tons. Assuming the ends to be position-fixed, what size of angle would be necessary?*

The required area, according to the parabolic formula, may be obtained directly by use of equation (498) for mild steel sections:

$$a = a_0 + \frac{c_2}{c_1} g L^2$$

where $g = \frac{a}{\kappa^2}$, $\frac{c_2}{c_1} = \frac{1}{30{,}000}$ and $c_1 = 40{,}000$ lb. sq. in. [equation (474)]. If $= 4$, $a_0$, the area required if the member were very short, would be

$$a_0 = \frac{4 \times 5 \times 2240}{40{,}000}.$$

A value for $g$ can be obtained from similar sections. For a 4 in. × 4 in. × ½ in. angle $g = \frac{3 \cdot 75}{(0 \cdot 77)^2} = 6 \cdot 3$. For a 4½ in. × 4½ in. × ½ in. angle $g = \frac{4 \cdot 25}{(0 \cdot 87)^2} = 5 \cdot 6$. Assume a mean value of 6. Then

$$a = \frac{4 \times 5 \times 2240}{40{,}000} + \frac{6 \times 60^2}{30{,}000}$$

$$= 1 \cdot 12 + 0 \cdot 72 = \mathbf{1 \cdot 84} \text{ sq. in.}$$

An angle 3 in. × 3 in. × ⅜ in., area = 2·11 sq. in., would appear to be suitable. For this, $g = \frac{2 \cdot 11}{(0 \cdot 58)^2} = 6 \cdot 3$, so that the area should really be $1 \cdot 12 + 0 \cdot 76 = 1 \cdot 88$ sq. in., and the section chosen is suitable. Checking the solution by the parabolic formula (474),

$$f_r = 40{,}000 - \frac{4}{3}\left(\frac{60}{\cdot 58}\right)^2 = 25{,}720 \text{ lb. sq. in.}$$

and the safe load $\quad W = \frac{25{,}720 \times 2 \cdot 11}{4 \times 2240} = \mathbf{6 \cdot 06}$ tons.

Attacking the problem next by the rational method (p. 248), from considerations of stability the minimum moment of inertia must be

$$I = \frac{5WL^2}{\pi^2 E} = \frac{5 \times 5 \times 60^2}{\pi^2 \times 13{,}000} = \mathbf{0 \cdot 71} \text{ in.}^4$$

Now the minimum I of a 3 in. × 3 in. × ⅜ in. angle = 0·72. Hence the section cannot be less than this, assume that this section is used. Then from equation (94) for the maximum stress

$$a = \frac{W}{f_c}\left\{1 + 1 \cdot 3 \frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\}.$$

Here $f_c = 5 \cdot 6$ tons sq. in. as in previous examples, $\frac{v_2}{\kappa^2} = \frac{1 \cdot 2}{(0 \cdot 58)^2}$, $\epsilon_1 = \frac{L}{750} = \frac{60}{750} = 0 \cdot 08$ in., and $\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1 \bar{v}_1}{a}$. Approximately $\bar{v}_1 = 0 \cdot 55$ and $\frac{a_1}{a} = \frac{1}{2}$. Hence, $\epsilon_2 = \frac{60}{1000} + \frac{1}{5} \times \frac{1}{2} \times 0 \cdot 55 = 0 \cdot 06 + 0 \cdot 06$, and $\epsilon_1 + \epsilon_2 = 0 \cdot 08 + 0 \cdot 12 = 0 \cdot 20$. Therefore $a = \frac{5}{5 \cdot 6}\left\{1 + 1 \cdot 3 \frac{1 \cdot 2}{(0 \cdot 58)^2} \times 0 \cdot 20\right\} = \mathbf{1 \cdot 72}$ sq. in. The area of a 3 in. × 3 in. × ⅜ in. angle is 2·11 sq. in., which is more than sufficient.

By both methods, therefore, it would appear that a 3 in. × 3 in. × ⅜ in.

angle is required. Checking this result by the Rankine-Gordon formula, the safe load is

$$W = \frac{1}{4} \cdot \frac{48{,}000 \times 2{\cdot}11}{2240\left\{1 + \frac{4}{30{,}000}\left(\frac{60}{0{\cdot}58}\right)^2\right\}} = \mathbf{4{\cdot}65} \text{ tons},$$

so that to give a factor of safety of 4 with this formula a somewhat larger area would be required.

*A column of angle section, 10 ft. long, is required to carry a load of 5 tons. Assuming the ends to be position-fixed, what size of angle would be necessary?*

In this example all the particulars are the same as in the last, except that the length is doubled. The direct solution for the area [equation (498)] becomes

$$a = a_0 + \frac{c_2}{c_1} gL^2 = \frac{4 \times 5 \times 2240}{4000} + \frac{6 \times 120^2}{30{,}000} = 1{\cdot}12 + 2{\cdot}88 = \mathbf{4} \text{ sq. in.},$$

from which it would appear that a $4\frac{1}{2}$ in. $\times$ $4\frac{1}{2}$ in. $\times$ $\frac{1}{2}$ in. angle is required. The minimum radius of gyration of this section is 0·87 in., and hence $\frac{L}{\kappa} = \frac{120}{0{\cdot}87} = 138$. Now the validity limit of equation (474) is $X_p = 122$. Hence the size of the section should be determined by Euler's formula. Adopting a minimum factor of safety of 5, the moment of inertia required is

$$I = \frac{5 \times 5 \times 120^2}{\pi^2 \times 13{,}000} = \mathbf{2{\cdot}81} \text{ in.}^4,$$

and hence a $4\frac{1}{2}$ in. $\times$ $4\frac{1}{2}$ in. $\times$ $\frac{7}{16}$ in. angle might be used ($I_{min} = 2{\cdot}83$). The tendency of the parabolic formula applied beyond its validity limit is to give too high a value for $a$.

Testing the stress in the $4\frac{1}{2}$ in. $\times$ $4\frac{1}{2}$ in. $\times$ $\frac{7}{16}$ in. angle by the stress formula, equation (94),

$$f_c = \frac{W}{a}\left\{1 + 1{\cdot}3\,\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\},$$

$$\frac{v_2}{\kappa_2} = \frac{1{\cdot}77}{(0{\cdot}87)^2},\quad a = 3{\cdot}74 \text{ sq. in.},\quad \epsilon_1 = \frac{L}{750} = \frac{120}{750} = 0{\cdot}16, \text{ and}$$

$$\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\,\frac{a_1\bar{v}_1}{a} = \frac{120}{1000} + \frac{1}{5} \times \frac{1}{2} \times 0{\cdot}9 = 0{\cdot}12 + 0{\cdot}09.$$

Hence $\epsilon_1 + \epsilon_2 = 0{\cdot}37$ in., and

$$f_c = \frac{5}{3{\cdot}74}\left\{1 + 1{\cdot}3\,\frac{1{\cdot}77}{(0{\cdot}87)^2} \times 0{\cdot}37\right\} = \mathbf{2{\cdot}85} \text{ tons sq. in.}$$

*A column of angle section 10 ft. long is required to carry a load of 5 tons. The ends are intended to be position- and direction-fixed. What size of angle is necessary?*

The particulars in this example are the same as those in the last except that the ends may be looked upon as imperfectly direction-fixed. Assuming a value $q = 0{\cdot}78$, the free length is $120 \times 0{\cdot}78 = 94$ in. The direct solution for the area becomes, therefore,

$$a = a_0 + \frac{c_2}{c_1} gL^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (498)$$

$$= \frac{4 \times 5 \times 2240}{40{,}000} + \frac{6 \times 94^2}{30{,}000} = 1{\cdot}12 + 1{\cdot}77 = \mathbf{2{\cdot}89} \text{ sq. in.}$$

A $3\frac{1}{2}$ in. $\times$ $3\frac{1}{2}$ in. $\times$ $\frac{1}{2}$ in. angle has an area of 3·25 sq. in., and a 4 in. $\times$ 4 in. $\times$ $\frac{7}{16}$ in. angle has an area of 3·30 sq. in. The latter is evidently the better section, the difference between the weights being negligible.

Attacking the problem by the rational method (p. 249), from considerations of stability the minimum value of the moment of inertia is

$$I = \frac{5W\,(qL)^2}{\pi^2 E} = \frac{5 \times 5 \times 94^2}{\pi^2 \times 13{,}000} = \mathbf{1{\cdot}72} \text{ in.}^4$$

A 4 in. $\times$ 4 in. $\times$ $\frac{3}{8}$ in. angle has a minimum I of 1·74 in.$^4$, and is therefore a possible solution. The minimum I of a $3\frac{1}{2}$ in. $\times$ $3\frac{1}{2}$ in. $\times$ $\frac{1}{2}$ in. angle is 1·5 in.$^4$, which is insufficient.

It is evident that the value of $\frac{W}{P_1}$ will be very nearly $\frac{1}{5}$, and hence equation (205A) for the stress at the centre of the column may be applied, from which

$$a = \frac{W}{f_c}\left\{1 + 0{\cdot}76\,\frac{\epsilon_1 v_2}{\kappa^2}\right\}.$$

The value of $f_c$ may be taken at 5·6 tons sq. in., $\frac{v_2}{\kappa^2} = \frac{1{\cdot}58}{(0{\cdot}78)^2}$, and for position- and direction-fixed columns $\epsilon_1 = \frac{L}{750} + \frac{1}{10}\frac{a_1\bar{v}_1}{a}$ [equation (422)]. Approximately $\frac{a_1}{a} = \frac{1}{2}$, $\bar{v}_1 = 0{\cdot}75$. Hence

$$\epsilon_1 = \frac{120}{750} + \frac{1}{10} \times \frac{1}{2} \times 0{\cdot}75 = 0{\cdot}16 + 0{\cdot}04 = 0{\cdot}20 \text{ in.}$$

Therefore

$$a = \frac{5}{5{\cdot}6}\left\{1 + 0{\cdot}76\,\frac{0{\cdot}20 \times 1{\cdot}58}{(0{\cdot}78)^2}\right\} = \mathbf{1{\cdot}25} \text{ sq. in.}$$

If the direction-fixing be perfect ($k = 1$), the maximum stress will occur at the ends of the column, and the requisite area, from equation (187), is

$$a = \frac{5}{5{\cdot}6}\left\{1 + 0{\cdot}78\,\frac{0{\cdot}20 \times 1{\cdot}58}{(0{\cdot}78)^2}\right\} = \mathbf{1{\cdot}26} \text{ sq. in.,}$$

or practically the same as before. The actual area of a 4 in. × 4 in. × $\frac{3}{8}$ in. angle is 2·86 sq. in., from which it would appear that this section is sufficiently strong.

It will be observed that by the rational method an angle 4 in. × 4 in. × $\frac{3}{8}$ in. may be used; whereas, according to the parabolic formula, a 4 in. × 4 in. × $\frac{7}{16}$ in. angle is necessary. On the other hand, according to the latter formula, a $3\frac{1}{2}$ in. × $3\frac{1}{2}$ in. × $\frac{1}{2}$ in. angle is suitable, whereas such a section would give a factor of safety of less than 5 with Euler's formula. It is evidently necessary, when working near the validity limit of the parabolic formula, to check the result by Euler's formula, or alternatively to use a factor of safety of 5 with the parabolic formula.

It is of interest to observe that the section 4 in. × 4 in. × $\frac{3}{8}$ in. determined by the rational method is worth 8·5 tons by the ordinary form of the Rankine-Gordon formula for position- and direction-fixed columns ($q = \frac{1}{4}$), or 5·2 tons if the free length 94 in. be used.

*A stanchion* 10 *ft. high, which may be considered as firmly built-in at its lower end, carries at its upper end an isolated load of* 30 *tons on a bracket such that the line of action of the load is* 9 *in. from the centre line of the stanchion* (Fig. 78). *What section of stanchion is required?*

If the column may be considered as direction-fixed at its lower end, it can be looked upon as part of an eccentrically loaded position-fixed column 20 ft. in length. The ordinary parabolic and Rankine-Gordon formulæ are no longer applicable here. The eccentricity form of the Rankine-Gordon formula may be used, but the rational method is better.

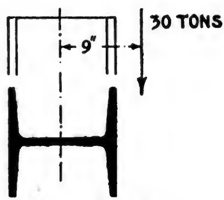



FIG. 78.

From conditions of stability the minimum value of the moment of inertia is

$$I = \frac{\eta W L^2}{\pi^2 E} = \frac{5 \times 30 \times 240^2}{\pi^2 \times 13{,}000} = 67\cdot3 \text{ in.}^4$$

The nearest convenient section fulfilling this condition is a broad flange beam $9\frac{1}{2}$ in. × $9\frac{1}{2}$ in. × 51 lb. per ft. The minimum I of this section is 73·1 in.$^4$

It is next necessary to try what area is required to carry the large bending moment set up by the eccentric load. For this purpose equation (94) may be applied:

$$a = \frac{W}{f_c}\left\{1 + 1\cdot3\,\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\}.$$

As before, $f_c = 5\cdot6$ tons sq. in. Assuming the broad flange beam to be a suitable section, $\frac{v_2}{\kappa^2} = 0\cdot29$, $\epsilon_1 = \frac{L}{750} = \frac{240}{750} = 0\cdot32$ in., and $\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a} = \frac{240}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a} =$ say 0·48 in. approximately, to which must be added

the intentional eccentricity of 9 in. Hence $(\epsilon_1 + \epsilon_2) = 0{\cdot}32 + 0{\cdot}48 + 9 = 9{\cdot}8$ in. Inserting these values in the formula,

$$a = \frac{30}{5{\cdot}6}\left\{1 + 1{\cdot}3 \times {\cdot}29 \times 9{\cdot}8\right\} = \mathbf{25{\cdot}2} \text{ sq. in.}$$

It is evident from this result that the $9\frac{1}{2} \times 9\frac{1}{2}$ beam section, of which the area is 15·0 sq. in., is too small. The area of a 12 in. × 12 in. × 80 lb. per ft. broad flange beam is 23·6 sq. in. Since, however, this cross section has larger dimensions as well as a greater area than that at first assumed, it will probably carry the load. Find the maximum stress by equation (94):

$$\epsilon_1 = \frac{L}{750} = 0{\cdot}32 \text{ in.} \qquad \epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a}.$$

For this section $\bar{v}_1 = 4{\cdot}9$ in., and $\frac{a_1}{a} = \frac{1}{2}$. Hence $\epsilon_2 = \frac{240}{1000} + \frac{1}{5} \times \frac{1}{2} \times 4{\cdot}9 + 9$ in. $= 0{\cdot}24 + 0{\cdot}49 + 9 = 9{\cdot}73$ in., and $(\epsilon_1 + \epsilon_2) = 0{\cdot}32 + 9{\cdot}73 = 10{\cdot}05$ in. The value of $\frac{v_2}{\kappa^2}$ for the section in the direction of the greatest moment of inertia is 0·23. Hence

$$f_c = \frac{30}{23{\cdot}6}\left\{1 + 1{\cdot}3 \times 0{\cdot}23 \times 10{\cdot}05\right\} = \mathbf{5{\cdot}1} \text{ tons sq. in.}$$

If, in addition, bending in a direction at right angles to the above be taken into account, $\epsilon_1 = 0{\cdot}32$, $\epsilon_2 = 0{\cdot}24 + 0{\cdot}23$, $(\epsilon_1 + \epsilon_2) = 0{\cdot}79$ in., and $\frac{v_2}{\kappa^2}$ in the direction of the least moment of inertia is 0·78. Therefore the maximum stress at one corner of the specimen is

$$f_c = \frac{W}{a}\left\{1 + 1{\cdot}3\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2) + 1{\cdot}3\frac{v_2'}{(\kappa')^2}(\epsilon_1' + \epsilon_2')\right\}$$

$$= \frac{30}{23{\cdot}6}\left\{1 + 1{\cdot}3 \times 0{\cdot}23 \times 10{\cdot}05 + 1{\cdot}3 \times 0{\cdot}78 \times 0{\cdot}79\right\}$$

$$= \mathbf{6{\cdot}1} \text{ tons sq. in.}$$

This stress exceeds the allowed limit, but since it is improbable that the column would suffer from the maximum imperfections in both directions at once, the section proposed might be adopted.

In the above problem the column has been assumed to carry an isolated load at the upper end, which has, therefore, been taken as free in both position and direction. In practice, however, it is probable that the member bringing the load on to the stanchion would fix the end of the latter, in one direction or the other, at least in position. On the other hand, it is improbable that the lower end of the column would be perfectly fixed in direction. Such a column, position-fixed at its upper end and imperfectly direction-fixed at its lower end, should be treated by the formulæ of Case IV, Variation 2.

*A position-fixed lattice-braced column of the type shown in Fig. 79 is 15 ft. long and carries a concentric load of 30 tons. The safe stress in the member, were it a tie, would be 5 tons sq. in. Design the column.*

Since the member is in compression, the given safe stress should be reduced by 20 per cent. to allow for the effect of initial stresses and shop treatment. Take the safe working stress, therefore, as $5 \times 0{\cdot}8 = 4$ tons sq. in.

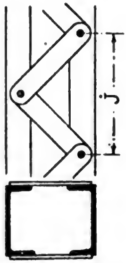


FIG. 79.

First assume that all the imperfections tend to produce bending in the plane $xz$. In this direction the column may be treated as a solid column. The minimum value of the moment of inertia of the column as a whole about the axis $yy$, if $\eta = 5$, is

$$I_z = \frac{5WL^2}{\pi^2 E} = \frac{5 \times 30 \times 180^2}{\pi^2 \times 13{,}000} = \mathbf{38}\ \text{in.}^4$$

The minimum value of the greatest moment of inertia of a single channel, therefore, is 19 in.$^4$

The value of $a_0$ is $\frac{30}{4} = 8$ sq. in. The value of $a$ will, of course, be greater than this, suppose it to be in the neighbourhood of 5 sq. in. for each individual channel. The nearest standard channel is a B.S.C. No. 9, 7 in. $\times$ 3 in. $\times$ 17·56 lb. per ft. For this section

$$a = 5{\cdot}16 \text{ sq. in.}$$
$$I_{max} = 37{\cdot}6 \text{ in.}^4 \qquad \kappa_{max} = 2{\cdot}70 \text{ in.}$$
$$I_{min} = 4{\cdot}01 \text{ in.}^4 \qquad \kappa_{min} = 0{\cdot}88 \text{ in.}$$

The required value of $a$ can be found from equation (94):

$$a = \frac{W}{f_c}\left\{1 + 1{\cdot}3\,\frac{v_2}{\kappa^2}(\epsilon_1 + \epsilon_2)\right\}.$$

Here $f_c = 4$ tons sq. in., $\frac{v_2}{\kappa^2} = \frac{3{\cdot}5}{2{\cdot}7^2} = 0{\cdot}48$.

$$\epsilon_1 = \frac{L}{750} = \frac{180}{750} = 0{\cdot}24 \text{ in.}$$

$$\epsilon_2 = \frac{L}{1000} + \frac{1}{5}\frac{a_1\bar{v}_1}{a} = \frac{180}{1000} + \frac{1}{5} \times \frac{2{\cdot}5}{2} = 0{\cdot}18 + 0{\cdot}25$$

for $\frac{a_1}{a} = \frac{1}{2}$, and $\bar{v}_1 = 2{\cdot}5$ in. approximately. Hence, $\epsilon_1 + \epsilon_2 = 0{\cdot}67$ in., and

$$a = \frac{30}{4}\left\{1 + 1{\cdot}3 \times 0{\cdot}48 \times 0{\cdot}67\right\} = \mathbf{10{\cdot}7} \text{ sq. in.}$$

The area of each individual channel should be, therefore, 5·35 sq. in., or rather greater than that assumed. Since, however, the value of I for the section as a

whole is nearly double the minimum value necessary for stability, it follows that the ratio of $\frac{W}{P_2}$ is nearly $\frac{1}{10}$ instead of $\frac{1}{5}$, and the more correct expression for $a$ [equation (93A)] becomes

$$a = \frac{30}{4}\left\{1 + 1{\cdot}15 \times 0{\cdot}48 \times 0{\cdot}67\right\} = \mathbf{10{\cdot}3} \text{ sq. in.}$$

from which the area of a single channel $= 5{\cdot}15$ sq. in. Unless the value of $h$ be made unduly large, however, the stresses due to bending in the plane $xy$ are likely to exceed those due to bending in the plane $xz$, and it would appear wiser to choose a slightly greater area, say a B.S.C. No. 12, 8 in. × 3 in. × 19·3 lb. per ft. For this section

$$\begin{aligned} a &= 5{\cdot}67 \text{ sq. in.} & v_2'' &= 2{\cdot}16 \text{ in.} \\ I_{max} &= 53{\cdot}4 \text{ in.}^4 & \kappa_{max} &= 3{\cdot}07 \text{ in.} \\ I_{min} &= 4{\cdot}33 \text{ in.}^4 & \kappa_{min} &= 0{\cdot}87 \text{ in.} \end{aligned}$$

Consider next bending in the plane $xy$. Since the value of $h$ is quite undetermined, it is necessary first to find what load an elementary flange column will safely carry. To this end preliminary values for $j$ and $F_c$ must be assumed. If the bracing be arranged as shown in Fig. 79, the value of $j$ will be something less than $2h$. Suppose that $j = 18$ in. $F_c$ will be slightly greater than $\frac{W}{2}$. Suppose that $F_c = 18$ tons. Then Euler's crippling load for the elementary flange column is

$$P'' = \frac{\pi^2 EI}{j^2} = \frac{\pi^2 \times 13{,}000 \times 4{\cdot}33}{18^2} = 1710 \text{ tons.}$$

The ratio $\frac{F_c}{P''} = \frac{18}{1710}$, and the factor $\left(1 + \frac{3F_c}{2P''}\right) = 1{\cdot}016$. Now, for the elementary column $\epsilon_1'' = \frac{j}{375} = \frac{18}{375} = 0{\cdot}05$, $\epsilon_2'' = \frac{j}{1000} + \frac{1}{5}\frac{a_1''\bar{v}_1''}{a_2} = \frac{18}{1000} + \frac{1}{5}\cdot\frac{2 \times 1}{5{\cdot}67} = 0{\cdot}02 + 0{\cdot}07$, for $a_1'' = 2$ sq. in., and $\bar{v}_1'' = 1$ in. approximately. Hence $\epsilon_1'' + \epsilon_2'' = 0{\cdot}14$ in., and $\frac{v_2''}{(\kappa'')^2} = \frac{2{\cdot}16}{0{\cdot}87^2} = 2{\cdot}86$. The maximum stress $f_c$ is given by equation (311),

$$f_c = \frac{F_c}{a_2}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\}$$

$$= \frac{18}{5{\cdot}67}\left\{1 + 2{\cdot}86 \times 1{\cdot}016 \times 0{\cdot}14\right\} = \mathbf{4{\cdot}47} \text{ tons sq. in.}$$

This stress exceeds the allowed limit, 4 tons sq. in. To reduce it to 4 tons sq. in., $F_c$ must be reduced to $\frac{18 \times 4}{4{\cdot}47} = \mathbf{16{\cdot}1}$ tons.

Substituting this value in equation (501),

$$\frac{L}{h} = \frac{75}{7}\left\{\frac{40F_c}{W} - 22{\cdot}25\right\} = \frac{75}{7}\left(\frac{40 \times 16{\cdot}1}{30} - 22{\cdot}25\right),$$

the value of $\frac{L}{h}$ becomes negative, and it is evident that the assumed value of $a_2$ is too small.

A new value for $a_2$ may be chosen and the process repeated, or, since nothing is fixed regarding $h$, a suitable value for D can be assumed, and hence the new value for $a_2$ found. Suppose that $D = 10$ in., then $h = 10 - 2 \times 0{\cdot}84 = 8{\cdot}32$ in. Therefore, for the column as a whole

$$I_y = \frac{ah^2}{4} = \frac{2 \times 5{\cdot}67 \times 8{\cdot}32^2}{4} = 196 \text{ in.}^4$$

$$P = \frac{\pi^2 EI_y}{L^2} = \frac{\pi^2 E \times 196}{180^2} = 777 \text{ tons,}$$

and the factor $\left(1 + \frac{3W}{2P}\right) = 1{\cdot}058$.

The value of $F_c$ is determined from equation (309):

$$F_c = W\left\{\frac{1}{2} + \left(1 + \frac{3W}{2P}\right)\frac{(\epsilon_1 + \epsilon_2)}{h}\right\}$$

Now $\epsilon_1 = \frac{L}{750} = \frac{180}{750} = 0{\cdot}24$ in., from equation (421) $\epsilon_2 = \frac{L}{1000} + \frac{h}{20} + \frac{h}{160}$ $= \frac{180}{1000} + \frac{8{\cdot}32}{20} + \frac{8{\cdot}32}{160}$, and $\epsilon_1 + \epsilon_2 = 0{\cdot}89$. Therefore,

$$F_c = 30\left\{\frac{1}{2} + 1{\cdot}058 \times \frac{0{\cdot}89}{8{\cdot}32}\right\} = \mathbf{18{\cdot}4} \text{ tons.}$$

The value of $a_2$ can be found from equation (311):

$$a_2 = \frac{F_c}{f_c}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\}.$$

The values of the factors have already been determined above, and

$$a_2 = \frac{18{\cdot}4}{4}\left\{1 + 2{\cdot}86 \times 1{\cdot}016 \times 0{\cdot}14\right\} = \mathbf{6{\cdot}5} \text{ sq. in.}$$

from which it would appear that a B.S.C. No. 13, 8 in. × 3½ in. × 22·7 lb. per ft. area 6·68 sq. in., is a suitable section; but it would probably be necessary to increase $h$ slightly, in order that the section might be more easily riveted up.

The maximum shearing force on the column is given by equation (445):

$$Q_{max} = \frac{\pi W}{2L}\left(\epsilon_2 + \frac{32\epsilon_1}{\pi^2}\right).$$

For bending in the plane $xy$, $\epsilon_2 = 0{\cdot}65$ and $\epsilon_1 = 0{\cdot}24$.

Hence $$Q_{max} = \frac{\pi \times 30}{2 \times 180}\left\{0{\cdot}65 + \frac{32 \times 0{\cdot}24}{\pi^2}\right\} = \mathbf{0{\cdot}38} \text{ ton.}$$

This is the maximum transverse shearing force on the column. There are (Fig. 79) two systems of lattice bracing to carry this load, but it is evident that the bars cannot be made sufficiently light.

*A bridge compression member of the ordinary type* (Fig. 77) *is firmly riveted to the top and bottom flanges of the main girder. Its length is* 10·5 *ft. The maximum and minimum loads on the member are* 106 *and* 49 *tons respectively. Find a suitable cross section. The inside width between the gussets at the ends of the column is* 1 *ft.* 8 *in.*

This is the problem common in bridge design. To allow for the varying load the Launhardt-Weyrauch formula will be adopted. For a tension member of mild steel this formula is

$$f_t = 5\left(1 + \frac{r}{2}\right) \text{ tons sq. in., where } r \text{ is the ratio } \frac{\text{minimum load}}{\text{maximum load}}.$$

Reducing this stress by 20 per cent. to allow for reductions in the quality of the material, the safe working stress in compression is

$$f_c = 4\left(1 + \frac{r}{2}\right) \text{ tons sq. in.}$$

In the present instance, $r = \dfrac{49}{106} = 0{\cdot}46$ and $f_c = 4\,(1 + 0{\cdot}23) = \mathbf{4{\cdot}9}$ tons sq. in.

The column will be imperfectly fixed in both directions, $q$ may be taken as 0·78, and $qL = 98$ in.

First assume that all the imperfections tend to produce bending in the plane $xz$. In this direction the column may be treated as a solid column. The minimum value of the moment of inertia of the column as a whole about the axis $yy$, if $\eta = 5$, is

$$I_z = \frac{5W(qL)^2}{\pi^2 E} = \frac{5 \times 106 \times 98^2}{\pi^2 E} = \mathbf{40} \text{ in.}^4$$

The minimum value of the greatest moment of inertia of a single channel, therefore, is 20 in.$^4$

The value of $a_0$ is $\dfrac{106}{4{\cdot}9} = 22$ sq. in. The value of $a$ will be greater than

this: try a section (Fig. 77) consisting of a B.S.C. No. 24, 12 in. × $3\frac{1}{2}$ in. × 26·1 lb. per ft., to which is riveted a 12 in. × $\frac{3}{8}$ in. plate. For this section

$$a = 12\cdot1 \text{ sq. in.} \qquad I_{max} = 212 \text{ in.}^4 \qquad I_{min} = 10\cdot5 \text{ in.}^4$$

Euler's crippling load for the column as a whole is

$$P_2 = \frac{4\pi^2 EI_z}{L^2} = \frac{4 \times \pi^2 \times E \times 424}{126^2} = 13{,}700 \text{ tons,}$$

the ratio $\dfrac{W}{P_2} = \dfrac{106}{13{,}700} = 0\cdot008$, and hence, from Fig. 15, the value of $k$ is 1·01. The true value of $a$ can be obtained from equation (181):

$$a = \frac{W}{f_c}\left[1 + \frac{2k\epsilon_1 v_2}{\kappa^2}\left\{0\cdot17 + 0\cdot26\frac{W}{P_2} + \frac{1}{\pi^2}\cdot\frac{P_2}{W}\left(1 - \frac{1}{k}\right)\right\}\right]$$

From equation (422), $\epsilon_1 = \dfrac{L}{750} + \dfrac{1}{10}\dfrac{a_1\bar{v}_1}{a} = \dfrac{126}{750} + \dfrac{1}{10} \times \dfrac{3\cdot7}{2} = 0\cdot353$ in.; for $\dfrac{a_1}{a} = \dfrac{1}{2}$ and $\bar{v}_1 = 1\cdot85$ in. The value of $\kappa^2$ is $\dfrac{212}{12\cdot1}$ and $v_2 = 6$ in. Hence $\dfrac{v_2}{\kappa^2} = 0\cdot34$.

Therefore $a = \dfrac{106}{4\cdot9}\Big[1 + 2 \times 1\cdot01 \times 0\cdot353 \times 0\cdot34 \times \cdot302\Big] = \mathbf{23\cdot3}$ sq. in.

Since, however, the stress may be greater at the ends than at the centre, it is necessary to find what area is required there. Taking in this case $k$ to be equal to unity, the worst possible assumption, the value of $a$ is found from equation (186)

$$a = \frac{W}{f_c}\left[1 + \frac{\epsilon_1 v_1}{\kappa^2}\left\{0\cdot66 + 0\cdot58\frac{W}{P_2}\right\}\right].$$

Here $v_1 = v_2$, and

$$a = \frac{106}{4\cdot9}\Big[1 + 0\cdot353 \times 0\cdot34 \times 0\cdot67\Big] = \mathbf{23\cdot4} \text{ sq. in.,}$$

and the section chosen ($a = 2 \times 12\cdot1$) is suitable.

Next consider bending in the plane $xy$. The dimension D is fixed at 1 ft. 8 in. Hence $h = 20 - 2 \times \cdot85 = 18\cdot3$ in., and for the column as a whole

$$I_y = \frac{ah^2}{4} = \frac{2 \times 12\cdot1 \times 18\cdot3^2}{4} = 2020 \text{ in.}^4$$

Therefore $$P_2 = \frac{4\pi^2 EI_y}{L^2} = \frac{4\pi^2 E \times 2020}{126^2} = 65{,}300 \text{ tons,}$$

and the ratio $\dfrac{W}{P_2}$ is so small that $k$ is practically equal to unity. It follows, therefore, that the maximum stress will occur at the ends of the column.

Giving to $k$ the value unity, the worst assumption, the maximum force $F_c$ on an elementary flange column is given by equation (331):

$$F_c = W\left[\frac{1}{2} + \frac{\epsilon_1}{h}\left\{0{\cdot}66 + 0{\cdot}58\frac{W}{P_2}\right\}\right].$$

Now, from equation (423),

$$\epsilon_1 = \frac{L}{750} + \frac{h}{40} = \frac{126}{750} + \frac{18{\cdot}3}{40} = 0{\cdot}63 \text{ in.}$$

Hence $$F_c = 106\left\{\frac{1}{2} + \frac{0{\cdot}63}{18{\cdot}3} \times 0{\cdot}67\right\} = \mathbf{55{\cdot}5} \text{ tons.}$$

Had the approximate equation (332) been used,

$$F_c = 106\left\{\frac{1}{2} + 0{\cdot}78 \times \frac{0{\cdot}63}{18{\cdot}3}\right\} = \mathbf{56} \text{ tons,}$$

which is nearly enough correct.

The load $55\frac{1}{2}$ tons is the load on an elementary flange column. If the bracing is at 45° and is of the double lattice type, $j$ will be about 16 in. Then Euler's crippling load for the elementary flange column is $P'' = \frac{\pi^2 EI}{j^2} = \frac{\pi^2 E \times 10{\cdot}5}{16^2} = 5{,}260$ tons, the ratio $\frac{F_c}{P''} = \frac{55{\cdot}5}{5260}$, and the factor $\left(1 + \frac{3F_c}{2P''}\right) = 1{\cdot}016$. Now for the elementary column $\epsilon_1'' = \frac{j}{375} = \frac{16}{375} = 0{\cdot}05$, $\epsilon_2'' = \frac{j}{1000} + \frac{1}{5}\frac{a_1''\bar{v}_1''}{a} = \frac{16}{1000} + \frac{1}{5} \times \frac{3 \times 1{\cdot}5}{12{\cdot}1} = 0{\cdot}02 + 0{\cdot}07$; for $a_1'' \doteq 3$ sq. in. and $\bar{v}_1'' = 1{\cdot}5''$ approximately. Hence $\epsilon_1'' + \epsilon_2'' = 0{\cdot}14$ in. If the flanges of the channel (Fig. 77) become the concave side of the elementary flange column $\frac{v_2''}{(\kappa'')^2} = \frac{3{\cdot}03}{(0{\cdot}93)^2} = 3{\cdot}5$, and the area necessary to carry the load of $55\frac{1}{2}$ tons is obtained from equation (311),

$$a_2 \doteq \frac{F_c}{f_c}\left\{1 + \frac{v_2''}{(\kappa'')^2}\left(1 + \frac{3F_c}{2P''}\right)(\epsilon_1'' + \epsilon_2'')\right\},$$

from which, since $f_c = 4{\cdot}9$ tons sq. in.,

$$a_2 = \frac{55{\cdot}5}{4{\cdot}9}\left\{1 + 3{\cdot}5 \times 1{\cdot}016 \times 0{\cdot}14\right\} = \mathbf{17} \text{ sq. in.}$$

Alternatively, if the flange plate become the concave side of the elementary flange column (Fig. 36), it is necessary to allow for the local buckling of the flange plate between the rivets. This plate is $\frac{3}{8}$ in. thick; and if $p$, the longitudinal pitch of the rivets, be limited to 16 times this, $p = 6''$. Hence, from equation (340),

$$f_c''' = \frac{f_c}{1 + \frac{p}{80t}} = \frac{4{\cdot}9}{1 + \frac{6}{80 \times \frac{3}{8}}} = \mathbf{4{\cdot}1} \text{ tons sq. in.}$$

If the flange plate become the concave surface, the value of $\frac{v_2''}{(\kappa'')^2} = \frac{0{\cdot}85}{(0{\cdot}93)^2}$ $= 0{\cdot}99$, and therefore, from equation (502),

$$a_2 = \frac{55{\cdot}5}{4{\cdot}1}\left\{1 + 0{\cdot}99 \times 1{\cdot}016 \times 0{\cdot}14\right\} = \mathbf{15{\cdot}5} \text{ sq. in.},$$

or less than the value obtained from equation (311). It is evident, however, that the provisional estimate $a_2 = 12{\cdot}1$ sq. in. is insufficient.

A section consisting of a B.S.C. No. 26, 12 in. $\times$ 4 in. $\times$ 36½ lb. per ft., to which is riveted a plate 12 in. $\times$ ½ in., will give an area of 16·7 sq. in.; which, in view of the increased value of $\frac{v_2''}{(\kappa'')^2}$ will be sufficient.

The maximum shearing force on the column is given by equation (447):

$$Q_{max} = W\frac{4k\epsilon_1}{L}$$

For bending in the plane $xy$, $k$ was seen to be practically equal to unity, and $\epsilon_1 = 0{\cdot}63$ in. Hence

$$Q_{max} = 106\,\frac{4 \times 1 \times 0{\cdot}63}{126} = \mathbf{2{\cdot}2} \text{ tons.}$$

This load will be divided between the four systems of diagonals, and in finding their cross section the variable nature of the load must be taken into account.

**Future Research.**—The most pressing point for future research on the subject of columns is undoubtedly the question of the degree of imperfection common in practical direction-fixed ends; in short, what value of $k$ should be assumed for such ends? A complete answer to this question is difficult, but at present the designer has *no real data whatsoever regarding practical end conditions.*

The problem of "secondary stresses" in framed structures is involved in this. In fact, it may well be found that the actual deflection curve of a column forming part of a framed structure is totally different from that assumed in the usual theory. Such evidence as there is points to S-shaped bending; and a new development of the column theory may grow out of secondary stress considerations, with the work of Winkler, Asimont, and Manderla for an underlying basis. A complete theory, however, must take into account the "give" of the riveted connexions between the members.

A second point is the question of secondary and tertiary flexure in practical built-up columns. What is the wave length of the secondary deflection curve? To what extent is it influenced by the point and method of attachment of the lattice bracing? In short, an answer to the questions on p. 180. To be of practical value the work must include the *combination* of the primary with the secondary and tertiary flexures.

A third point is the question of the best material for columns, particularly for large bridge compression members. With this is bound up the question of the employment of nickel steel in columns.

A fourth point is the question of the most suitable form of cross section for large bridge compression members.

In connexion with the development of the aeroplane, new strut problems will without doubt arise. No attempt is made to include such here.

## ADDENDA

### *From Part I. Historical—see Preface*

LAMARLE'S value (1846) for the maximum deflection of an ideal position-fixed column is:

$$\dot{y}_0 = 4\left(1 - \frac{W}{Ea}\right)\sqrt{\frac{S}{W}}\left\{\frac{L}{\pi\sqrt{1 - \frac{W}{Ea}}}\sqrt{\frac{W}{S}} - 1\right\}^{\frac{1}{2}}$$

KROHN'S formula (1908) for built-up mild-steel columns (see p. 187) is:

$$F_e = W\left(\frac{1}{2} + \frac{y_0}{h}\right) = W\frac{a_2}{a}\,\frac{272}{272 - \frac{L}{\kappa}}.$$

$$\text{If } a_2 = \frac{a}{2}, \text{ and } \kappa = \frac{h}{2}, \text{ then}$$

$$F_e = W\,\frac{68h}{136h - L}.$$

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